จำนวนจีโอดีทิคของกราฟโซ่เชิงเส้น

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บทคัดย่อ

สำหรับจุดยอด u และ v ในกราฟ G ช่วงปิดของ $I[u,v]$ คือเซตที่ประกอบด้วยจุดยอด u,v และทุก ๆ จุดยอดที่อยู่ในบางวิถีสั้นสุด $u-v$ ในกราฟ G สำหรับ $S \subseteq V(G)$ โดยที่ $S \neq \varnothing$ ช่วงปิดของ $I[S]$ คือยูเนียนของช่วงปิด $I[u,v]$ โดยที่ $u,v \in S$ จะเรียกเซต S ว่า เซตจีโอดีทิคของกราฟ G ถ้า $I[S]=V(G)$ และเซตจีโอดีทิคที่มีขนาดเล็กที่สุดของกราฟ G เรียกว่า เซตจีโอดีทิคที่เล็กสุดของกราฟ G และจำนวนสมาชิกของเซตจีโอดีทิคที่เล็กสุด คือ จำนวนจีโอดีทิคของกราฟ G เขียนแทนด้วย $g(G)$ สำหรับกราฟกำหนดทิศทาง D จำนวนจีโอดีทิคของกราฟ D สามารถนิยามได้ในลักษณะเดียวกันกับในกราฟเชิงเดียว จำนวนจีโอดีทิคกำหนดทิศทางล่างของกราฟ G เขียนแทนด้วย $g^{-}(G)$ คือค่าน้อยสุดของจำนวนจีโอดีทิคของกราฟที่เกิดจากการกำหนดทิศทางของกราฟ G จำนวนจีโอดีทิคกำหนดทิศทางบนของกราฟ G เขียนแทนด้วย $g^{+}(G)$ คือค่ามากสุดของจำนวนจีโอดีทิคของกราฟที่เกิดจากการกำหนดทิศทางของกราฟ G ในงานวิจัยนี้ได้ศึกษาจำนวนจีโอดีทิคของกราฟโซ่เชิงเส้นและศึกษาจำนวนจีโอดีทิคกำหนดทิศทางล่างและจำนวนจีโอดีทิคกำหนดทิศทางบนของกราฟโซ่เชิงเส้น

คำสำคัญ: กราฟโซ่เชิงเส้น/ จำนวนจีโอดีทิค/ จำนวนจีโอดีทิคกำหนดทิศทางล่าง/ จำนวนจีโอดีทิคกำหนดทิศทางบน

The Geodetic Number of Linear Chain Graphs

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Abstract

For two vertices u and v of a graph G, the closed interval $I[u,v]$ is a set of all vertices lying on some $u-v$ geodesic in G. For a nonempty subset S of $V(G)$, the set $I[S]$ is the union of all closed interval $I[u,v]$, where $u,v \in S$. A set S is called a geodetic set of G if $I[S]=V(G)$. A geodetic set having minimum cardinality is called a minimum geodetic set of G and this cardinality is called the geodetic number of G, which is denoted by $g(G)$. For a digraph D, the geodetic number of D can be defined in a similar manner. The lower orientable geodetic number $g^{-}(G)$ of G is the minimum geodetic number of an orientation of G. The upper orientable geodetic number $g^{+}(G)$ of G is the maximum geodetic number of an orientation of G. In this work, we verify the geodetic number of linear chain graphs and study the lower and upper orientable geodetic numbers of some linear chain graphs. Moreover, an upper bound of $g^{+}(G)$ is investigated.

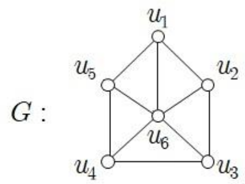
Keywords: Linear chain graph/ Geodetic number/ Lower orientable geodetic number/ Upper orientable geodetic number

Introduction

For two vertices u and v of a connected graph G, a shortest $u-v$ path in G is called a $u-v$ geodesic. We define the *closed interval* of two vertices u and v to be the set of u,v and all vertices lying on some $u-v$ geodesic in G, and is denoted by $I_G[u,v]$ or simply $I[u,v]$ if the graph G under consideration is clear. For a nonempty subset S of $V(G)$, we define

$$I[S] = \bigcup_{u,v\in S} I[u,v] \tag{1.1}$$

A set S of vertices of G is called a *geodetic set* of G if $I[S]=V(G)$. A geodetic set of minimum cardinality is called a *minimum geodetic set* and this cardinality is the *geodetic number* of G, which is denoted by $g(G)$. To illustrate this idea, consider the graph G of Figure 1.



Let $S_1 = \{u_1,u_3,u_4,u_6\}$ The closed intervals of every pair of two distinct vertices in S_1 are $I[u_1,u_3] = \{u_1,u_2,u_3,u_6\}$, $I[u_1,u_4] = \{u_1,u_4,u_5,u_6\}$, $I[u_1,u_6] = \{u_1,u_6\}$, $I[u_3,u_4] = \{u_3,u_4\}$, $I[u_3,u_6] = \{u_3,u_6\}$ and $I[u_4,u_6] = \{u_4,u_6\}$. Therefore, $I[S_1]=V(G)$, that is, S_1 is a geodetic set of G However, S_1 is not a minimum geodetic set of G since the set $S_2 = \{u_2,u_4,u_5\}$ is also a geodetic set of G such that $|S_2| < |S_1|$.For each pair u and v of vertices of G, since the closed interval $I[u,v] \neq V(G)$, it follows that G contains no geodetic set of cardinality 2. This implies that $g(G) = 3$.

The concept of geodetic number in graphs was introduced by Harary, Loukakis and Tsouros in [1] that determining the geodetic number of a graph is an NP-hard problem. Chartrand and Zhang in [2] presented the forcing geodetic number of a graph by investigating a subset of minimum geodetic set. The geodetic number of graphs has been studied further in [3,4].

Materials and methods

In this research, we investigate the geodetic number of linear chain graphs and verify the lower and upper orientable geodetic numbers of some linear chain graphs by using theorems as follows:

Theorem 2.1 ([5])**.** A graph G has an antidirected orientation if and only if G is bipartite.

Theorem 2.2 ([5])**.** Let D be a nontrivial oriented graph of order n. Then $g(D) = n$ if and only if D is transitive.

Theorem 2.3 ([5])**.** Let G be a connected graph of order $n \geq 2$. If G contains a Hamiltonian path, then $g^{-}(G) = 2$.

Theorem 2.4 ([3])**.** If G is a connected bipartite graph of order $n \geq 2$, then $g^{+}(G) = n$.

Results

1. The geodetic number of linear chain graphs

Let G be a plane graph. A *face* of G is an induced subgraph of G that is a cycle. If a face F of G is an even cycle, then F is called an *even face* of G. Similarly, if a face F of G is an odd cycle, then F is called an *odd face* of G. For two faces F_1 and F_2 in G, if uv is the only one edge in F_1 and F_2, then F_1 and F_2 are said to be *adjacent* in G, and the edge uv and vertices u, v are called the *shared edge* and *shared vertices* of F_1 and F_2, respectively. For a positive integer n, a *chain graph* G_n is a connected plane graph consisting of n faces $F_1, F_2, \ldots, F_n$ such that F_i is adjacent to F_{i+1} for each integer i with $1 \leq i \leq n-1$. Let G_n be a chain graph with n faces $F_1, F_2, \ldots, F_n$ such that $F_i = C_{n_i} = (v_{i,1}, v_{i,2}, \ldots, v_{i,n_i}, v_{i,1})$. G_n is said to be a *linear chain graph* if $v_{j,\left\lceil \frac{n_j}{2} \right\rceil} = v_{j+1,1}$ and $v_{j,\left\lceil \frac{n_j}{2} \right\rceil+1} = v_{j+1,n_{j+1}}$ for each integer j with $1 \le j \le n-1$. For example, the graph G_5 of Figure 2 is a linear chain graph with five faces: $F_1 = C_6, F_2 = C_6, F_3 = C_7, F_4 = C_8$ and $F_5 = C_5$. For

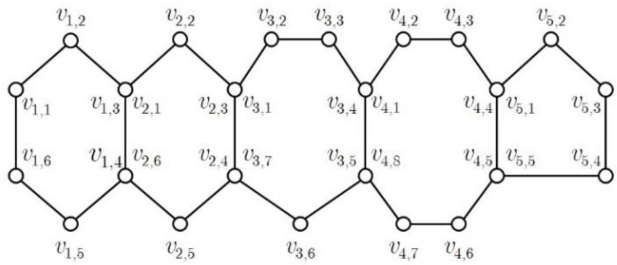


Figure 2 A linear chain graph G_5 with 5 faces

each integer i with $1 \le i \le n$, the vertices $v_{i,1}, v_{i,2}, \ldots, v_{i,\left\lceil \frac{n_i}{2} \right\rceil}$ are called *upper vertices* of G_n and $v_{i,n_i}, v_{i,n_{i-1}}, \ldots, v_{i,\left\lceil \frac{n_i}{2}+1 \right\rceil}$ are called *lower vertices* of G_n. An *even linear chain graph* E_r is a linear chain graph with r even faces. An *odd linear chain graph* O_s is a linear chain graph with s odd faces. An even linear chain graph E_5 with 5 faces $F_1 = C_8, F_2 = C_4, F_3 = C_8, F_4 = C_6$ and $F_5 = C_4$, and an odd linear chain graph O_6 with 6 faces $F_1 = C_7$, $F_2 = C_5, F_3 = C_5, F_4 = C_9, F_5 = C_7$ and $F_6 = C_5$ are shown in Figure 3 (a) and (b), respectively. We first verify the geodetic number of an even linear chain graph.

Theorem 1.1. For a positive integer r, let E_r be an even linear chain graph. Then $g(E_r) = 2$.

Proof. Let E_r be an even linear chain graph consisting of r faces $F_1, F_2, \ldots, F_r$ such that $F_i = C_{r_i} = (u_{i,1}, u_{i,2}, \ldots, u_{i,r_i}, u_{i,1})$ for each integer i with $1 \le i \le r$. We verify that $S = \{u_{1,1}, u_{r,\frac{r_r}{2}+1}\}$ is a minimum geodetic set of E_r.

Since a $u_{1,1} - u_{r,\frac{r}{2}+1}$ path that its interior vertices are upper vertices of E_r and $u_{1,1} - u_{r,\frac{r}{2}+1}$ path that its interior vertices are lower vertices of E_r, are $u_{1,1} - u_{r,\frac{r}{2}+1}$ geodesic, it follows that $I[u_{1,1}, u_{r,\frac{r}{2}+1}] = V(E_r)$, that is, S is a geodetic set of E_r. Since E_r is not trivial, it follows that S is a minimum geodetic set of E_r and so $g(E_r) = 2$.

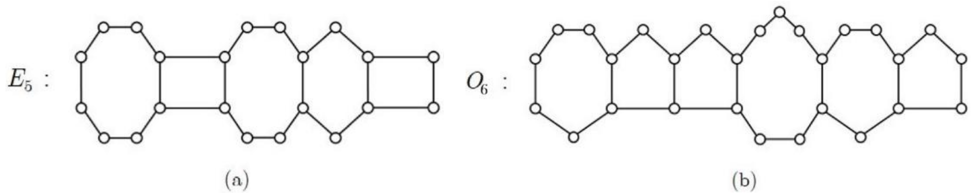


Figure 3 An even linear chain graph E_5 and odd linear chain graph O_6

To show the geodetic number of an odd linear chain graph, we verify the useful result.

Theorem 1.2. For an integer $s \geq 3$, let O_s be an odd linear chain graph with s faces $F_1, F_2, \ldots, F_s$ such that $F_i = C_{s_i} = (v_{i,1}, v_{i,2}, \ldots, v_{i,s_i}, v_{i,1})$, where $1 \leq i \leq s$. For integer j with $2 \leq j \leq s-1$, if $x \in \{v_{j-1,1}, v_{j-1,s_{j-1}}\}$ and $y \in \{v_{j+1,\lceil \frac{s_{j+1}}{2} \rceil}, v_{j+1,\lceil \frac{s_{j+1}}{2} \rceil+1}\}$, then every $x-y$ geodesic contains no upper vertices that are not shared vertices, of F_j .

Proof. Assume, to the contrary, that there is an $x-y$ geodesic P of length l in O_s containing $v_{j,k}$ for some integer k with $2 \leq k \leq \lceil \frac{s_j}{2} \rceil - 1$ .Since $v_{j,k}$ has degree 2, it follows that P contains $v_{j,k-1}$ and $v_{j,k+1}$ .This implies that P must also contain all upper vertices of F_j . By a case-by-case analysis, it can be shown that there is an $x-y$ path of length less than l, contradicting the fact that P is an $x-y$ geodesic of length l. Hence, every $x-y$ geodesic contains no upper vertices that are not shared vertices, of F_j .

The following corollary is a consequence of Theorem 3.1.2.

Corollary 1.3. For an integer $s \geq 3$, let O_s be an odd linear chain graph with s faces $F_1, F_2, \ldots, F_s$ such that $F_i = C_{s_i} = (v_{i,1}, v_{i,2}, \ldots, v_{i,s_i}, v_{i,1})$, where $1 \leq i \leq s$ and let $W = (V(F_{j-1}) \cup V(F_j) \cup V(F_{j+1})) - \{v_{j-1,1}, v_{j-1,s_{j-1}}, v_{j+1,\lceil \frac{s_{j+1}}{2} \rceil}, v_{j+1,\lceil \frac{s_{j+1}}{2} \rceil+1}\}$ for some integer j with $2 \leq j \leq s-1$ .if $S \subseteq V(O_s) - W$, then S is not a geodetic set of O_s.

As we know, the geodetic number of an odd cycle is 3. We therefore verify the geodetic number of an odd linear chain graph with at least 2 faces.

Theorem 1.4. For an integer $s \geq 2$, let O_s be an odd linear chain graph with s faces. Then $g(O_s) = \lfloor \frac{s}{3} \rfloor + 2$.

Proof. Let O_s be an odd linear chain graph with s faces $F_1, F_2, \ldots, F_s$ such that $F_i = C_{s_i} = (v_{i,1}, v_{i,2}, \ldots, v_{i,s_i}, v_{i,1})$, where $1 \leq i \leq s$. We claim that the set $S = \{v_{3j,\lfloor \frac{s_{3j}}{2} \rfloor} \mid j = 1, 2, \ldots, \lfloor \frac{s}{3} \rfloor\} \cup \{v_{1,1}, v_{s,\lceil \frac{s_s}{2} \rceil}\}$ is a minimum geodetic set of O_s. It is routine to verify that $I[S] = V(G)$ and so S is a geodetic set of O_s. To show that the cardinality of S is minimum, we may assume, to the contrary, that there is a geodetic set S' of O_s having cardinality $\lfloor \frac{s}{3} \rfloor + 1$. Since O_s contains no geodetic set of cardinality 1, it follows that $s \geq 3$. Therefore, S' contains at least two vertices, one belonging to F_1 and one belonging to F_s. For otherwise, there is no $u-v$ geodesic in O_s containing vertices of F_1 and F_s, where $u, v \in S'$. Let x and y be vertices of S' belonging to F_1 and F_s, respectively. We consider two cases.

Case 1. $x \in \{v_{1,1}, v_{1,s_1}\}$ and $y \in \{v_{s,\lceil \frac{s_s}{2} \rceil}, v_{s,\lceil \frac{s_s}{2} \rceil+1}\}$.

Since O_s has s faces and $\left|S' - \{x,y\}\right| = \left\lfloor \frac{s}{3} \right\rfloor - 1$, it follows that there are three consecutive faces $F_{i'-1}, F_{i'}, F_{i'+1}$ of O_s $(2 \le i' \le s-1)$ such that $S' \subseteq V(O_s) - W$, where $W = (V(F_{i'-1}) \cup V(F_{i'}) \cup V(F_{i'+1})) - \{v_{i'-1,1}, v_{i'-1,s_{i'-1}}, v_{i'+1,\left\lceil \frac{s_{i'+1}}{2} \right\rceil}, v_{i'+1,\left\lceil \frac{s_{i'+1}}{2} \right\rceil+1}\}$. Thus, S' is not a geodetic set of O_s by Corollary 3.1.3, which is a contradiction.

Case 2. $x \notin \{v_{1,1}, v_{1,s_1}\}$ or $y \notin \{v_{s,\left\lceil \frac{s_s}{2} \right\rceil}, v_{s,\left\lceil \frac{s_s}{2} \right\rceil+1}\}$.

We may assume that $x \notin \{v_{1,1}, v_{1,s_1}\}$. Then S' contains at least two vertices of F_1. For otherwise, there is no $u-v$ geodesic in O_s containing $v_{1,1}$, where $u,v \in S'$. Thus, there are at most $\left\lfloor \frac{s}{3} \right\rfloor - 2$ vertices belonging to faces $F_2, F_3, \ldots, F_{s-1}$. Therefore, there are three consecutive faces $F_{i'-1}, F_{i'}, F_{i'+1}$ of O_s $(3 \le i' \le s-1)$ such that $S' \subseteq V(O_s) - W$, where
$W = (V(F_{i'-1}) \cup V(F_{i'}) \cup V(F_{i'+1})) - \{v_{i'-1,1}, v_{i'-1,s_{i'-1}}, v_{i'+1,\left\lceil \frac{s_{i'+1}}{2} \right\rceil}, v_{i'+1,\left\lceil \frac{s_{i'+1}}{2} \right\rceil+1}\}$. By Corollary 3.1.3, S' is not a geodetic set of O_s. This is also a contradiction. Therefore, O_s contains no geodetic set of cardinality $\left\lfloor \frac{s}{3} \right\rfloor + 1$. Hence, S is a minimum geodetic set of O_s and so $g(O_s) = \left\lfloor \frac{s}{3} \right\rfloor + 2$.

In order to show the geodetic number of a linear chain graph, we need an additional definition. For each integer i with $1 \le i \le n$, let G_n be a linear chain graph with n faces $F_1, F_2, \ldots, F_n$ such that $F_i = C_{n_i} = (v_{i,1}, v_{i,2}, \ldots, v_{i,n_i}, v_{i,1})$. For an integer j with $2 \le j \le s-1$,let $G_{n-1}^{F_j}$ be the graph obtained from $G_n - V(F_j)$ by adding new edge uv and joining both $v_{j-1,\left\lceil \frac{n_{j-1}}{2} \right\rceil-1}$ and $v_{j+1,2}$ to u and joining both $v_{j-1,\left\lceil \frac{n_{j-1}}{2} \right\rceil+2}$ and $v_{j+1,n_{j+1}-1}$ to v. We can say that $G_{n-1}^{F_j}$ is the graph obtained by removing the face F_j of G_n and joining faces F_{j-1} and F_{j+1} by a new shared edge. More generally, if X is a set of k faces of G_n except F_1 and F_n, then G_{n-k}^X is the graph obtained by removing all faces in X of G_n and joining the remaining faces by at most k new shared edges.

Theorem 1.5. For a positive integer n, let G_n be a linear chain graph with n faces $F_1, F_2, \ldots, F_n$ such that G_n contains an even face F_i for some integer i with $2 \le i \le n-1$ and let S be a minimum geodetic set of $G_{n-1}^{F_i}$. If two new shared vertices u and v of $G_{n-1}^{F_i}$ do not belong to S, then S is also a minimum geodetic set of G_n.

Proof. Let uv be the new shared edge of $G_{n-1}^{F_i}$ such that u is an upper vertex of $G_{n-1}^{F_i}$ and v is a lower vertex of $G_{n-1}^{F_i}$. Assume that S contains no u and v. Since S is a minimum geodetic set of $G_{n-1}^{F_i}$, it follows that there are four vertices $x_1, x_2, y_1, y_2 \in S$ such that an $x_1 - y_1$ geodesic in $G_{n-1}^{F_i}$ contains u, say P and $x_2 - y_2$ geodesic in $G_{n-1}^{F_i}$ contains v, say Q. We first verify that S is a geodetic set of G_n. In order to do this, we consider two cases.

Case 1. P contains v or Q contains u.

We may assume, without loss of generality, that P contains v. Let u' and v' be vertices of P that are adjacent to u and v, respectively, and let P' and P'' be $x_1 - u'$ and $v' - y_1$ subpaths of P, respectively. Since S contains no u and v, it follows that $S \subseteq V(G_n)$. Since F_i is an even face of G_n, two paths $P_1 = (v_{i,1}, v_{i,2}, \ldots, v_{i,\frac{n_i}{2}+1})$ and $P_2 = (v_{i,1}, v_{i,n_i}, \ldots, v_{i,\frac{n_i}{2}+1})$ are $v_{i,1} - v_{i,\frac{n_i}{2}+1}$ geodesics in G_n. This implies that two $x_1 - y_1$ paths, one obtained by following P' by P_1 and following P_1 by P'', and one obtained by following P' by P_2 and following P_2 by P'' are $x_1 - y_1$ geodesic in G_n .Therefore $I_{G_n}[x_1, y_1]$ contains all vertices of F_i. For every vertex $w \in V(G_n) - V(F_i)$, there are two vertices $x, y \in S$ such that w belongs to an $x - y$ geodesic in

$G_{n-1}^{F_i}$. Therefore, w is also in $x-y$ geodesic in G_n, that is, S is a geodetic set of G_n.

Case 2. P contains no v and Q contains no u.
Let u' and u'' be vertices of P that are adjacent to u and let v' and v'' be vertices of Q that are adjacent to v. We may assume that the x_1-u subpath of P contains u', say P', and $u-y_1$ subpath of P contains u'', say P''. Similarly, suppose that x_2-v and $v-y_2$ subpaths of Q contain v' and v'', respectively. Since S contains no u and v, it follows that $S \subseteq V(G_n)$. Let $P_1=(v_{i,1},v_{i,2},\ldots,v_{i,\frac{n_i}{2}})$ and $Q_1=(v_{i,n_i},v_{i,n_i-1},\ldots,v_{i,\frac{n_i}{2}+1})$ be paths of upper and lower vertices of F_i, respectively. Thus, the path x_1-y_1 obtained by following P' by P_1 and following P_1 by P'' is an x_1-y_1 geodesic in G_n and the path x_2-y_2 obtained by following Q' by Q_1 and following Q_1 by Q'' is also x_2-y_2 geodesic in G_n. This implies that $I_{G_n}[x_1,y_1]\cup I_{G_n}[x_2,y_2]$ contains all vertices of F_i. Similar to the proof of case 1, every vertex in $V(G_n)-V(F_i)$ is in $I_{G_n}[x,y]$ for some vertices $x,y\in S$. Hence, S is a geodetic set of G_n. Next, we show that S is a minimum geodetic set of G_n. Assume, to the contrary, that G_n contains geodetic sets of cardinality at most $|S|-1$. Among all geodetic sets of cardinality at most $|S|-1$, let S' be one with $S'\subseteq V(G_n)-V(F_i)$. Then there are four vertices $x_1,x_2,y_1,y_2\in S'$ such that $I_{G_n}[x_1,y_1]$ and $I_{G_n}[x_2,y_2]$ contain upper vertices and lower vertices of F_i, respectively. Therefore, $I_{G_{n-1}^{F_i}}[x_1,y_1]$ and $I_{G_{n-1}^{F_i}}[x_2,y_2]$ are also contain u and v of $G_{n-1}^{F_i}$, respectively. Let $w\in V(G_{n-1}^{F_i})-\{u,v\}$. Then w belongs to G_n. Since S' is a geodetic set of G_n, it follows that $w\in I_{G_n}[x,y]$ for some vertices $x,y\in S'$,that is $w\in I_{G_{n-1}^{F_i}}[x,y]$. This implies

that S' is a geodetic set of $G_{n-1}^{F_i}$, contradicting the fact that S is a minimum geodetic set of $G_{n-1}^{F_i}$. Hence, S is a minimum geodetic set of G_n.

The following corollary is a consequence of Theorem 3.1.5.

Corollary 1.6. For a positive integer n, let G_n be a linear chain graph with n faces $F_1, F_2, \ldots, F_n$ and let X be a set of all even faces of G_n except F_1 and F_n. If S is a minimum geodetic set of $G_{n-|X|}^{X}$ that contains no new shared vertices, then S is also a minimum geodetic set of G_n.

The geodetic number of a linear chain graph is presented in terms of the geodetic number of even and odd linear chain graphs, as we show next.

Theorem 1.7. For a positive integer n, let G_n be a linear chain graph with n faces. Then

(i) $g(G_n) = g(E_n) = 2$ if G_n contains no odd face,

(ii) $g(G_n) = g(O_s) = \left\lfloor \frac{s}{3} \right\rfloor + 2$ if G_n contains s odd faces, where $1 \le s \le n$.

Proof. (i) If G_n contains no odd face, then G_n is a even linear chain graph. Thus, $g(G_n) = g(E_n) = 2$ by Theorem 3.1.1.

(ii) Assume that G_n contains s odd faces, where $1 \le s \le n$. Then G_n contains $n-s$ even faces. We consider three cases.

Case 1. Both F_1 and F_n are odd faces of G_n.

Let X be a set of all even faces of G_n. Then $G_s^X = O_s$ and so $g(O_s) = \left\lfloor \frac{s}{3} \right\rfloor + 2$ by Theorem 3.1.4. Therefore, there is a minimum geodetic set S of O_s having cardinality $\left\lfloor \frac{s}{3} \right\rfloor + 2$. This implies by Corollary 3.1.6 that S is also a minimum geodetic set of G_n, that is, $g(G_n) = g(O_s) = \left\lfloor \frac{s}{3} \right\rfloor + 2$.

Case 2. Either F_1 or F_n (but not both) is an odd face of G_n.

We may assume that F_1 is an odd face and F_n is an even face of G_n. Let X be a set of all even faces of G_n except F_n. Then $G^X_{n-|X|}$ consists of s consecutive odd faces and one even face. Therefore, we may assume that G^X_{s+1} is a linear chain graph with $s+1$ faces $F_i' = C_{n_i} = (v_{i,1}, v_{i,2}, \ldots, v_{i,n_i}, v_{i,1})$ for each integer i with $1 \le i \le s+1$. It is routine to show that the set $S = \{v_{3j,\left\lfloor \frac{s_{3j}}{2} \right\rfloor} \mid j = 1, 2, \ldots, \left\lfloor \frac{s}{3} \right\rfloor\} \cup \{v_{1,1}, v_{s+1,\frac{(s+1)_{(s+1)}}{2}}\}$ is a minimum geodetic set of G^X_{s+1} and so S is also a minimum geodetic set of G_n by Corollary 3.1.6. Thus, $g(G_n) = g(O_s) = \left\lfloor \frac{s}{3} \right\rfloor + 2$.

Case 3. Neither F_1 nor F_n is an odd face of G_n.

By proceeding in the same manner as Case 2, we can show that $g(G_n) = g(O_s) = \left\lfloor \frac{s}{3} \right\rfloor + 2$.

2. The lower and upper orientable geodetic numbers

Let D be an oriented graph and let u and v be vertices of D. A shortest directed $u-v$ path in D is called a $u-v$ geodesic in D. We define $I[u,v]$ as the set of all vertices lying on either $u-v$ geodesic or $v-u$ geodesic in D. For a nonempty subset S of $V(D)$, we define $I[S]$ as in (1.1). A set S of D is called a *geodetic set* of D if $I[S] = V(D)$. A geodetic set of minimum cardinality is also referred to as a *minimum geodetic set* of D. The *geodetic number* $g(D)$ of D is the cardinality of a minimum geodetic set of D. For a vertex v of an oriented graph D, notice that if v has indegree 0 or outdegree 0, then v is either the initial or the terminal vertex of a geodesic containing v. This implies that every geodetic set of D must contain all vertices v such that $\mathrm{id}\, v = 0$ or $\mathrm{od}\, v = 0$. In fact, every geodetic set of D must contain all of its end-vertices. An oriented graph D is *antidirected* if every vertex of D has indegree 0 or outdegree 0. An oriented graph D is *transitive* if whenever (u,v) and (v,w) are arcs of D, then (u,w) is an arc of D. For a connected graph G of order $n \ge 2$, the *lower orientable geodetic* *number* $g^{-}(G)$ of G is defined as the minimum geodetic number of an orientation of G and the *upper orientable geodetic number* $g^{+}(G)$ of G as the maximum such geodetic number, that is,

$$g^{-}(G) = \min \{g(D) \mid D \text{ is an orientation of } G\},$$

$$g^{+}(G) = \max \{g(D) \mid D \text{ is an orientation of } G\}.$$

Therefore, every connected graph G of order $n \geq 2$ satisfies the following.

$$2 \leq g^{-}(G) \leq g^{+}(G) \leq n$$

The concept of the lower and upper orientable geodetic numbers of an oriented graph were presented by Chartrand and Zhang in [6] that the lower and upper orientable geodetic numbers of a graph are not the same. Moreover, the lower and upper orientable geodetic numbers and the several well known classes of graphs were investigated. Kim in [4] gave a partial answer for the conjecture by Chartrand and Zhang. Some results on orientable geodetic number were investigated. The lower and upper orientable geodetic numbers of graphs have been studied further in [3, 5, 7].

In this section, we verify the lower and upper orientable geodetic numbers of some linear chain graphs. Using Theorem 2.3, we are able to verify the lower orientable geodetic number of a linear chain graph. We omit the proof of this theorem since it is straightforward.

Theorem 2.1. For a positive integer n, let G_n be a linear chain graph with n faces. Then $g^{-}(G_n) = 2$.

Next, we determine the upper orientable geodetic number of an even linear chain graph.

Theorem 2.2. For a positive integer r, let E_r be an even linear chain graph with r faces. Then $g^{+}(E_r) = |V(E_r)|$.

Proof. Since E_r contains no odd cycle, it follows that E_r is bipartite. By Theorem 2.4, we obtain that $g^{+}(E_r) = |V(E_r)|$.

In order to show the upper orientable geodetic number of an odd linear chain graph, we first present a useful lemma.

Lemma 2.3. Let G be a connected graph of order n that contains no triangle. If G contains k disjoint odd cycles, then $g^{+}(G) \le n - k$.

Proof. Let $G_1, G_2, \ldots, G_k$ be k disjoint cycles of order at least 5 in G and let D be an orientation of G. For each integer i with $1 \le i \le k$, since G_i is not bipartite, it follows by Theorem 2.1 that G_i has no an antidirected orientation, that is, there is a vertex v_i in G_i such that $\mathrm{id}_D\, v_i > 0$ and $\mathrm{od}_D\, v_i > 0$. Thus, there are two vertices $u_i, w_i \in V(G_i)$ such that $(u_i, v_i), (v_i, w_i) \in V(G_i)$. Since G contains no triangle, it follows that v_i lies on $u_i - w_i$ geodesic. Clearly, $S = V(D) - \{v_1, v_2, \ldots, v_k\}$ is a geodetic set of D and so every minimum geodetic set of D contains at most $|S| = n - k$ vertices. This implies that $g(D) \le n - k$, that is, $g^{+}(G) \le n - k$.

We are now prepared to determine the upper orientable geodetic number of an odd linear chain graph.

Theorem 2.4. For a positive integer s, let O_s be an odd linear chain graph with s faces. Then $g^{+}(O_s) = |V(O_s)| - \lceil \frac{s}{2} \rceil$.

Proof. It is obvious that O_s contains $\lceil \frac{s}{2} \rceil$ disjoint odd cycles. Then, by Lemma 3.2.3, we see that $g^{+}(O_s) \le |V(O_s)| - \lceil \frac{s}{2} \rceil$. We next verify that $g^{+}(O_s) \ge |V(O_s)| - \lceil \frac{s}{2} \rceil$ by showing that there is an orientation D of O_s such that $|V(O_s)| - \lceil \frac{s}{2} \rceil$ vertices have indegree 0 or outdegree 0. In order to do this, we first consider two paths. Let $P = (v_{1,1} = u_1, u_2, \ldots, u_p = v_{s,\lceil \frac{s_s}{2} \rceil})$ be the $v_{1,1} - v_{s,\lceil \frac{s_s}{2} \rceil}$ path containing all upper vertices of O_s and let $Q = (v_{1,s_1} = w_1, w_2, \ldots, w_q = v_{s,\lceil \frac{s_s}{2} \rceil + 1})$ be the $v_{1,s_1} - v_{s,\lceil \frac{s_s}{2} \rceil + 1}$ path containing all lower vertices of O_s. Notice that $p = q + s$. We construct an orientation D

of O_s by (i) for integer i with $1 \le i \le \left\lfloor \frac{p-1}{2} \right\rfloor$, we direct the edges of P from u_{2i} toward u_{2i-1}, u_{2i+1} and if p is even, then we direct the edge $u_{p-1}u_p$ from u_p toward u_{p-1}, (ii) for integer j with $1 \le j \le \left\lfloor \frac{q-1}{2} \right\rfloor$, we direct the edges of Q from w_{2j-1}, w_{2j+1} toward w_{2j} and if q is even, then we direct the edge w_{q-1}, w_q from w_{q-1} toward w_q, (iii) for each adjacent vertices $u_{i'}$ and $w_{j'}$ $(1 \le i' \le p$ and $1 \le j' \le q)$ of O_s, if both i' and j' are even or both i' and j' are odd, then we direct the remaining edges of O_s from $w_{j'}$ toward $u_{i'}$ when $\mathrm{id}_Q\ w_{j'} = 0$ and from $u_{i'}$ toward $w_{j'}$ when $\mathrm{od}_Q\ w_{j'} = 0$; while if either i' or j' (but not both) is even, then we direct the remaining edges of O_s from $w_{j'}$ toward $u_{i'}$. By the constructing of D, there are $\left\lceil \frac{s}{2} \right\rceil$ vertices such that their indegree and outdegree are not 0. Therefore, D contains $|V(O_s)| - \left\lceil \frac{s}{2} \right\rceil$ vertices having indegree or outdegree 0. This implies that a minimum geodetic set of D must contains at least $|V(O_s)| - \left\lceil \frac{s}{2} \right\rceil$ vertices. Thus, $g(D) \ge |V(O_s)| - \left\lceil \frac{s}{2} \right\rceil$ and so $g^+(O_s) \ge |V(O_s)| - \left\lceil \frac{s}{2} \right\rceil$. Hence, $g(O_s) = |V(O_s)| - \left\lceil \frac{s}{2} \right\rceil$.

Discussion

For verifying the geodetic number of linear chain graphs by using the definition, we must investigate a minimum geodetic set S of linear chain graphs for showing that the geodetic number of linear chain graphs is the cardinality of S. Indeed, the results of this research are useful for verifying the geodetic number of linear chain graphs including the lower and upper orientable geodetic numbers of some linear chain graphs as shown in this research.

Conclusion

The geodetic number of a linear chain graph is presented in terms of the geodetic number of even and odd linear chain graphs are shown in Theorem 1.7, as we state next.

Theorem 1.7. For a positive integer n, let G_n be a linear chain graph with n faces. Then

(i) $g(G_n) = g(E_n) = 2$ if G_n contains no odd face,

(ii) $g(G_n) = g(O_s) = \left\lfloor \frac{s}{3} \right\rfloor + 2$ if G_n contains s odd faces, where $1 \le s \le n$.

Next, the lower orientable geodetic number of a linear chain graph is shown in Theorem 2.1, as we state next.

Theorem 2.1. For a positive integer n, let G_n be a linear chain graph with n faces. Then $g^-(G_n) = 2$.

The upper orientable geodetic number of some linear chain graphs is shown in Theorem 2.2 and 2.4 as follows.

Theorem 2.2. For a positive integer r, let E_r be an even linear chain graph with r faces. Then $g^+(E_r) = |V(E_r)|$.

Theorem 2.4. For a positive integer s, let O_s be an odd linear chain graph with s faces. Then $g^+(O_s) = |V(O_s)| - \left\lceil \frac{s}{2} \right\rceil$.

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