

Numerical Approximations of Fredholm-Volterra Integral Equation of 2nd kind using Galerkin and Collocation Methods

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Abstract

Galerkin and collocation approximation techniques are very effective and popular among researchers for numerical approximations of different types of differential, integral and integro-differential equations. Both methods approximate the solution by a finite sum of some known polynomials. In recent years, researchers around the world have been used different combinations of polynomials and collocation points in Galerkin and collocation methods for numerical approximations of different types of integral equations. Also, collocation method have been used more frequently compared to the Galerkin method. In this research, five different polynomials in Galerkin method and five different combinations of polynomials and collocation points in collocation method have been used for numerical approximations of linear FVIE of 2nd kind. It is found that the performances of different polynomials and collocation points in both these methods are consistent.

Keywords: Fredholm-Volterra integral equations, Galerkin method, collocation method, polynomials.

1. Introduction

There are mainly two classes of integral equations: Fredholm integral equation and Volterra integral equation. Both of these have linear and nonlinear forms. In this study, linear Fredholm-Volterra integral equations of 2nd kind are considered. Moreover, a similar study on Fredholm integral equation of 2nd kind is carried out by Molla and Saha, 2018.

There are several analytic methods and many approximation techniques available to solve different variations of integral equations. Among various approximation techniques, Galerkin and collocation methods are most popular and efficient, and they are also used to solve different versions of differential and integro-differential equations. In both Galerkin and collocation methods, unknown function is approximated by a finite sum of a set of known functions called as basis functions and such choice can be made from a wide variety of polynomials. Also, both of these methods follow different approaches to determine the expansion coefficients.

Fredholm-Volterra integral equations can be reduced to a system of algebraic equations by both Galerkin and collocation techniques. Yousuf and Razzaghi, 2005 used Legendre wavelets as the basis

function in spectral method and then used zeroes of Chebyshev polynomials of first kind as the collocation points to solve nonlinear Fredholm-Volterra integral equations. Then Mandal and Bhattacharya, 2007 used Galerkin technique with Bernstein polynomials as basis functions for numerical approximate solutions of some classes of integral equations. Recently Hesameddini and Shahbazi, 2017 used Bernstein polynomials in spectral collocation method with Gauss-Legendre nodes as collocation points to solve system of Fredholm-Volterra integral equations. Shifted Chebyshev polynomials of 1st kind and roots of shifted Chebyshev polynomials of 1st kind are considered as basis and collocation points respectively by Dastjerdi and Ghaini, 2012 to solve linear FVIE. But at first, they transformed the FVIE by moving least square method and then applied spectral approximations. Wang and Wang, (2013, 2014) first transformed system of Fredholm-Volterra integral equations into matrix equations by collocation scheme where they used Lagrange's basis polynomials as the basis functions in the approximate solution. Later, they used Taylor polynomials as the basis in spectral method and

equally spaced nodes as collocation points and hence transformed the FVIE to matrix equation. Ebrahimi and Rashidinia, 2015 introduced a method for linear and nonlinear Fredholm and Volterra integral equations where approximate solution is collocated by cubic B-spline. Nemati, 2015 presented spectral method based on shifted Legendre polynomials and shifted Gauss Legendre nodes as collocation points for numerical approximation of the solution of Fredholm-Volterra integral equation. Also, Fibonacci polynomials and equally spaced nodes are used by Mirzaee and Hoseini, 2016 in spectral collocation method. More recently Liu et al., 2018 proposed a new spectral collocation technique where they used modified weighted Lagrange function known as barycentric Lagrange interpolation function along with Gauss-Lobatto nodes to solve linear high-dimensional Fredholm integral equations.

As far as our knowledge is concern, Galerkin method is used less frequently for Fredholm-Volterra integral equations and Bernstein polynomials are only used as basis functions. Then in collocation method, roots of Jacobi polynomials have not been considered yet for FVIE of 2nd kind.

In this study, Galerkin and collocation methods are considered for approximations of numerical solutions of linear Fredholm-Volterra integral equation of type II. Galerkin method is applied with five different polynomials: Legendre, Chebyshev 1st kind, Bernstein, Lagrange's and Fibonacci polynomials to observe their performance in numerical approximations of linear FVIE of type II. Numerical examples are used and results from each polynomial are compared with the available exact solution. Then collocation method is also applied with five different combinations of basis functions and collocation points and details are presented in Table 1. And hence the approximations are again compared with exact solution in each case. In each case collocation points are shifted in the required interval according to the numerical example.

Remaining portions of this article is presented as follows: In Section 1.1, brief introduction of polynomials and sets of collocation points are given. Then, details of formulation of system of linear algebraic equations from linear FVIE of 2nd kind using both Galerkin and collocation methods are given in Section 1.2. After that, numerical results of both the methods using different polynomials and collocation points are compared and resultant

absolute errors are illustrated graphically in Section 1.3. Finally, in Section 1.4, a conclusion about this research is drawn.

Table 1. Combinations of polynomials and collocation points

Set of basis function	Collocation points
Chebyshev polynomials of 1 st kind	Roots of Chebyshev polynomials of 1 st kind
Legendre polynomials	Roots of Legendre polynomials
Bernstein polynomials	Roots of Jacobi polynomials
Lagrange's basis polynomials	Equally spaced nodes
Fibonacci polynomials	Gauss-Lobatto nodes

1.1 Introduction of polynomials and collocation points

In this section a very short introduction of Legendre, Chebyshev, Bernstein, Lagrange's, Fibonacci and Jacobi polynomials are given. All these polynomials are being used frequently in approximations of the solution of various kinds of differential and integral equations. Techniques to generate collocation points for arbitrary interval $[a, b]$ from the roots of Legendre, Chebyshev and Jacobi polynomials; from the Gauss-Lobatto nodes and equally spaced nodes are presented.

Legendre polynomials: Legendre polynomials $P_i(w)$ are set of orthogonal polynomials over $[-1, 1]$ and are solutions of the Legendre differential equation.

We know that explicit formula for $P_i(w)$ is

$$P_i(w) = \sum_{k=0}^i \binom{i}{k} \binom{-i-1}{k} \left(\frac{1-w}{2}\right)^k, i = 0, 1, 2, \dots$$

And, the recurrence relation for Legendre polynomials are as follows:

$$\begin{aligned} P_0(w) &= 1, P_1(w) = w \\ (i+1)P_{i+1}(w) &= (2i+1)wP_i(w) - iP_{i-1}(w), \\ & i = 1, 2, \dots \end{aligned}$$

The roots of the Legendre polynomials are in the interval $(-1, 1)$. In order to generate set of collocation points

$$LCP = [x_k]_{k=0}^{k=n}$$

over the interval $[a, b]$, the following points along with $x_0 = a$ and $x_n = b$ are considered.

$$x_k = \frac{b-a}{2} + \frac{b-a}{2} w_k; k = 1, 2, \dots, n-1$$

where w_k is a root of the Legendre polynomials $P_{n-1}(w)$ with $w_k < w_{k+1}$ for $k = 1, 2, \dots, n-2$.

Chebyshev polynomials: Chebyshev polynomials of first kind $T_i(y)$ are set of orthogonal polynomials over $[-1,1]$ and are solutions of the Chebyshev differential equation. We know that explicit formula for $T_i(y)$ is

$$T_i(y) = y^i \sum_{k=0}^{\lfloor \frac{i}{2} \rfloor} \binom{i}{2k} (1 - y^{-2})^k, i = 0,1,2, \dots$$

And, the recurrence relation for Chebyshev polynomials of first kind are as follows:

$$T_0(y) = 1, T_1(y) = y \\ T_{i+1}(y) = 2yT_i(y) - T_{i-1}(y), i = 1,2, \dots$$

Like Legendre polynomials, roots of the Chebyshev polynomials are in the interval $(-1,1)$. In order to generate set of collocation points

$$CCP = [x_k]_{k=0}^{k=n}$$

over the interval $[a, b]$, following points along with $x_0 = a$ and $x_n = b$ are considered.

$$x_k = \frac{b-a}{2} + \frac{b-a}{2} y_k ; k = 1,2, \dots, n-1$$

where y_k are the roots of the Chebyshev polynomials $T_{n-1}(y)$ with $y_k < y_{k+1}$ for $k = 1, 2, \dots, n-2$.

Bernstein polynomials: The i^{th} degree Bernstein polynomials defined on the interval $[a, b]$ are

$$B_{r,i}(y) = \binom{i}{r} \frac{(y-a)^r (b-y)^{i-r}}{(b-a)^i} ; a \leq y \leq b, \\ r = 0,1,2, \dots, i$$

where

$$\binom{i}{r} = \frac{i!}{r!(i-r)!}$$

There are $(i+1)$ Bernstein polynomials of i^{th} degree with following properties:

$$B_{r,i}(y) = 0, \text{ if } r < 0 \text{ or } r > i \\ B_{r,i}(a) = B_{r,i}(b) = 0, 1 \leq r \leq i-1$$

Lagrange basis polynomials: With $(n+1)$ points $x_0, x_1, x_2, \dots, x_{n-1}, x_n$, Lagrange basis polynomials $L_i(x); i = 0,1, 2, \dots, n$ are defined by

$$lp(x) = \prod_{r=0}^n (x - x_r) \\ L_i(x) = \frac{lp(x)}{lp'(x)(x - x_i)} ; i = 0,1,2, \dots, n$$

with the property $L_i(x_j) = \delta_{ij}$, where δ_{ij} is the Kronecker delta function. Here $lp'(x)$ is the derivative of $lp(x)$.

Fibonacci polynomials: Fibonacci polynomials are set of polynomials $F_i(u)$ defined by

$$F_i(u) = \sum_{k=0}^{\lfloor \frac{i}{2} \rfloor} \binom{i-k}{k} u^{i-2k}, i = 0,1,2, \dots$$

And, the recurrence relation for Fibonacci polynomials are follows:

$$F_0(u) = 1, F_1(u) = u \\ F_{i+1}(u) = uF_i(u) + F_{i-1}(u), i = 1,2,3, \dots$$

Jacobi polynomials: Jacobi polynomials $P_i^{\alpha,\beta}(y)$ are set of orthogonal polynomials on the interval $[-1,1]$ with respect to the weight function $(1-y)^\alpha(1+y)^\beta$. Roots of Jacobi polynomials are lies in the interval $(-1,1)$. Hence, to generate set of collocation points

$$JCP = [x_k]_{k=0}^{k=n}$$

over the interval $[a, b]$, the following points along with $x_0 = a$ and $x_n = b$ are considered:

$$x_k = \frac{b-a}{2} + \frac{b-a}{2} y_k ; k = 1,2, \dots, n-1$$

where y_k are the roots of the Jacobi polynomials $P_{n-1}^{\alpha,\beta}(y)$ with $y_k < y_{k+1}$ for $k = 1,2, \dots, n-2$ and

$$P_i^{\alpha,\beta}(y) = \frac{\Gamma(\alpha+i+1)}{i! \Gamma(\alpha+\beta+i+1)} \\ \sum_{m=0}^i \binom{i}{m} \frac{\Gamma(\alpha+\beta+i+m+1)}{\Gamma(\alpha+m+1)} \left(\frac{y-1}{2}\right)^m$$

Gauss-Lobatto nodes: In order to generate set of collocation points

$$GLCP = [x_k]_{k=0}^{k=n}$$

over the interval $[a, b]$, let us consider:

$$x_k = \frac{b-a}{2} + \frac{b-a}{2} \cos \frac{k\pi}{n} ; k = 0,1,2, \dots, n$$

Equally spaced nodes: In order to generate set of collocation points

$$ESCP = [x_k]_{k=0}^{k=n}$$

from equally spaced nodes over the interval $[a, b]$, following points are considered:

$$x_k = a + k \frac{b-a}{n} ; k = 0,2, \dots, n$$

1.2 Solution procedure for linear FVIE

All integral equations can describe various physical phenomena, scientific and engineering problems. Linear Fredholm-Volterra integral equations of 2nd kind arises from different types of

differential equations and can be described in many physical phenomena which have scientific interests. Here we consider the following general form of linear Fredholm-Volterra integral equation of 2nd kind:

$$\phi(x) + \lambda_1 \int_a^b k_1(x,t)\phi(t)dt + \lambda_2 \int_a^x k_2(x,t)\phi(t)dt = f(x), \quad a \leq x \leq b \quad (1)$$

where $k_1(x,t), k_2(x,t)$ and $f(x)$ are known functions, λ_1 and λ_2 are known parameters, and $\phi(x)$ is the unknown solution of Eq. (1), needed to be resolved.

To approximate the solution $\phi(x)$ of Eq. (1) using Galerkin and collocation methods, first let's, consider the following form of the trial solution $\tilde{\phi}(x)$:

$$\tilde{\phi}(x) = \sum_{i=0}^n m_i Q_i(x) \quad (2)$$

Here $Q_i(x)$ are called as basis functions and generally some known polynomials are used as basis function. In trial solution, m_i are the unknown parameters also known as expansion coefficients. Using the trial solution from Eq. (2) into Eq. (1), we can have

$$\begin{aligned} \sum_{i=0}^n m_i Q_i(x) + \lambda_1 \int_a^b k_1(x,t) \sum_{i=0}^n m_i Q_i(t) dt \\ + \lambda_2 \int_a^x k_2(x,t) \sum_{i=0}^n m_i Q_i(t) dt \\ = f(x) \\ \Rightarrow \sum_{i=0}^n m_i \left[Q_i(x) + \lambda_1 \int_a^b k_1(t,x) Q_i(t) dt \right. \\ \left. + \lambda_2 \int_a^x k_2(t,x) Q_i(t) dt \right] \\ = f(x) \end{aligned} \quad (3)$$

To determine the expansion coefficients by Galerkin method, first multiply Eq. (3) by $Q_j(x)$ and then integrate with respect to x from a to b . Thus Eq. (3) reduces to

$$\begin{aligned} \sum_{i=0}^n m_i \left[\int_a^b \left[Q_i(x) + \lambda_1 \int_a^b k_1(t,x) Q_i(t) dt \right. \right. \\ \left. \left. + \lambda_2 \int_a^x k_2(t,x) Q_i(t) dt \right] Q_j(x) dx \right] \\ = \int_a^b f(x) Q_j(x) dx; \quad j = 0, 1, \dots, n \end{aligned}$$

This is equivalent to the following linear system of equations:

$$\sum_{i=0}^n m_i R_{i,j} = F_j, \quad j = 0, 1, 2, \dots, n \quad (4)$$

where

$$\begin{aligned} R_{i,j} \\ = \int_a^b \left[Q_i(x) + \lambda_1 \int_a^b k_1(t,x) Q_i(t) dt \right. \\ \left. + \lambda_2 \int_a^x k_2(t,x) Q_i(t) dt \right] Q_j(x) dx; \\ F_j = \int_a^b f(x) Q_j(x) dx, \quad i, j = 0, 1, 2, \dots, n \end{aligned}$$

System of linear equations in Eq. (4) is called the Galerkin equation and by solving Eq. (4), the expansion coefficients m_i can be determined easily.

In collocation method, to determine expansion coefficients m_j , chose a point x_j in the domain for each m_j in the trial solution. These points x_j are known as collocation points. Then forcing Eq. (3) to satisfy at each x_j yields

$$\begin{aligned} \sum_{i=0}^n m_i \left[Q_i(x_j) + \lambda_1 \int_a^b k_1(t,x_j) Q_i(t) dt \right. \\ \left. + \lambda_2 \int_a^{x_j} k_2(t,x_j) Q_i(t) dt \right] \\ = f(x_j); \end{aligned} \quad (5)$$

$j = 0, 1, 2, \dots, n$

Thus, a trial solution with $(n + 1)$ unknown parameters produce following linear system of equations:

$$\sum_{i=0}^n m_i G_{i,j} = H_j; \quad j = 0, 1, 2, \dots, n \quad (6)$$

where

$$G_{i,j} = Q_i(x_j) + \lambda_1 \int_a^b k_1(t, x_j) Q_i(t) dt + \lambda_2 \int_a^x k_2(t, x_j) Q_i(t) dt ;$$

$$H_j = f(x_j), \quad i, j = 0, 1, 2, \dots, n$$

System of linear equations in Eq. (6) is called the collocation equation and by solving Eq. (6), the expansion coefficients m_i can be determined. So, once the set of basis functions $\{Q_i\}_{i=0}^n$ is chosen then based on Galerkin or collocation approach, expansion coefficients m_i can be found using Eqs. (4) or (6). And then substituting these values into Eq. (2) will give the approximate solution of FVIE of type II in terms of finite sum of series of functions $\{Q_i\}_{i=0}^n$.

1.3 Results and discussion

This research is performed to compare the performance of different polynomials in Galerkin method and different combinations of polynomials and sets of collocation points in collocation method for numerical approximations of linear Fredholm-Volterra integral equation of 2nd kind. Followings are the list of polynomials and combinations used for these two methods:

Table 2. Sets of basis and collocation points for both methods

Galerkin		Collocation
Set of basis	Set of basis	Set of collocation points
$\{P_i\}_{i=0}^n$	$\{P_i\}_{i=0}^n$	LCP
$\{T_i\}_{i=0}^n$	$\{T_i\}_{i=0}^n$	CCP
$\{B_{i,n}\}_{i=0}^n$	$\{B_{i,n}\}_{i=0}^n$	JCP
$\{L_i\}_{i=0}^n$	$\{L_i\}_{i=0}^n$	ESCP
$\{F_i\}_{i=0}^n$	$\{F_i\}_{i=0}^n$	GLCP

With $n = 5$, five sets of collocation points for the interval $[0,1]$ are as follows:

Table 3. Sets of collocation points for $n = 5$

LCP	{0, 0.0694318, 0.330009, 0.669991, 0.930568, 1}
CCP	{0, 0.0380602, 0.308658, 0.691342, 0.96194, 1}
JCP	{0, 0.117472, 0.357384, 0.642616, 0.882528, 1}
ESCP	{0, 0.2, 0.4, 0.6, 0.8, 1}
GLCP	{0, 0.0954915, 0.345492, 0.654508, 0.904508, 1}

Here two numerical examples are considered to carry out the investigations and in both examples domain of the problem is $[0,1]$.

Example 1: Consider the following example of linear FVIE of 2nd kind:

$$\phi(x) + \int_0^1 e^{x+t} \phi(t) dt - \int_0^x e^{x+t} \phi(t) dt = e^{-x} - (x-1)e^x, \quad 0 \leq x \leq 1$$

Exact solution of this problem is $\phi(x) = e^{-x}$.

At first, the performance of Legendre, Chebyshev, Bernstein, Lagrange's and Fibonacci polynomials are observed in Galerkin method for this problem. Absolute errors of example 1 in Galerkin method for five sets of basis functions given in Table 2 with $n = 5$ are presented in Fig. 1.

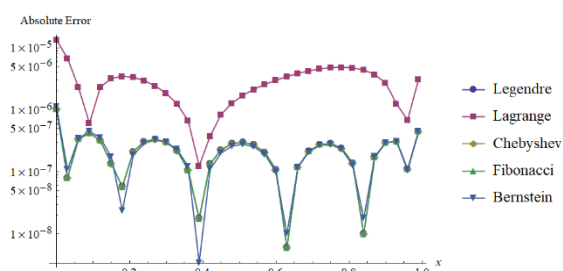


Figure 1: Absolute error curves of example 1 in Galerkin method

Now, performance of different combinations of polynomials and set of collocation points are observed in collocation method. Numerical solutions of example 1 in collocation method with different combinations given in Table 2 with $n = 5$ are given in the Table 4 and corresponding absolute error curves are presented in Fig. 2.

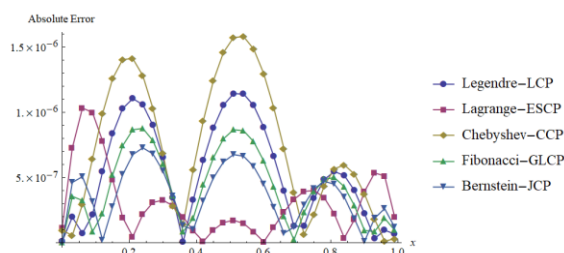


Figure 2: Absolute error graphs of example 1 in collocation method

Example 2: Consider the following example of linear FVIE of 2nd kind used by Wang and Wang, 2013:

$$\phi(x) + \int_0^1 e^t \sin x \phi(t) dt - \int_0^x e^t \cos x \phi(t) dt = f(x), \quad 0 \leq x \leq 1$$

where

$$f(x) = e^x - \frac{1}{2}(e^{2x} - 1) \cos x + \frac{1}{2}(e^2 - 1) \sin x$$

Exact solution of this problem is $\phi(x) = e^x$

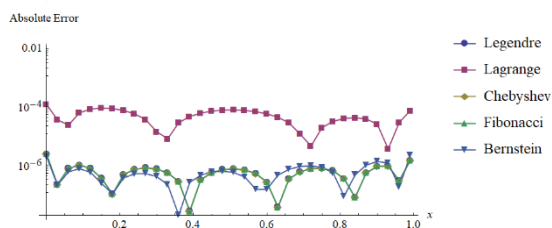


Figure 3: Absolute error curves of example 2 in Galerkin method

Like the previous example, at first, the performance of Legendre, Chebyshev, Bernstein, Lagrange's and Fibonacci polynomials are observed in Galerkin method for this problem. Absolute errors of example 2 in Galerkin method for five sets of basis given in Table 2 with $n = 5$ are presented in Fig. 3.

Now, performance of different combinations of polynomials and set of collocation points are observed in collocation method. Numerical solutions of example 2 in collocation method with different combinations given in Table 2 with $n = 5$ are also given in Table 5 and corresponding absolute error curves are presented in Fig. 4.

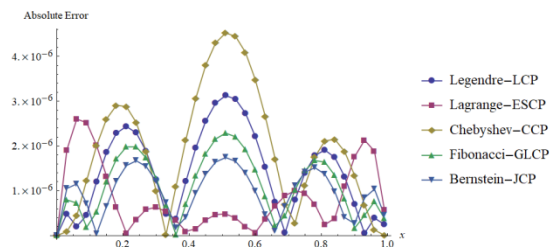


Figure 4: Absolute error graphs of example 2 in collocation method

After investigation of the absolute errors of both examples in Galerkin method, it is evident that Legendre, Chebyshev, Fibonacci and Bernstein polynomials give better solution than Lagrange's polynomial whereas solutions from first three polynomials are almost the same. Though there are some variations between solutions from Bernstein and Legendre, Chebyshev and Fibonacci but no clear conclusion can be made about the performance of the polynomials. In collocation method, performance of different combinations of basis functions and collocation point's sets are consistent in both problems. Lagrange's polynomials with ESCP handle the errors better than the others except around the both boundaries. Among rest of the combinations, overall performance of Bernstein &

JCP, Fibonacci & GLCP, Legendry & LCP and Chebyshev & CCP follows the downward trend.

1.4 Conclusion

In this article, formulation of system of linear algebraic equations for linear FVIE of 2nd kind by both Galerkin and collocation methods are presented to determine the expansion coefficients. Then five different polynomials: Legendre, Chebyshev, Fibonacci, Bernstein and Lagrange's are being used in Galerkin method. In both test problems, it is found that first four polynomials give better solution than Lagrange's polynomials. It is also noticed that in Galerkin method, Legendre, Chebyshev and Fibonacci polynomials give the same solutions. In collocation method, five different combinations of polynomials and collocation points are being tried. All the combinations produced very good approximations. Lagrange's polynomials and ESCP performed well compared to the others followed by Bernstein and JCP and performance of Chebyshev polynomials and CCP are worst.

Conflict of Interest

The authors do not report any financial or personal connections with other persons or organizations, which might negatively affect the contents of this publication and/or claim authorship rights to this publication.

Publication Ethic

Submitted manuscripts must not have been previously published by or be under review by another print or online journal or source.

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Appendix: A

Table 4. Solutions of example 1 in collocation method

x	Exact	Legendre-LCP	Chebyshev-CCP	Bernstein-JCP	Lagrange-ESCP	Fibonacci-GLCP
0.	1.000000000	0.999999986	0.999999903	0.999999998	0.999999881	1.000000005
0.1	0.904837418	0.904837092	0.904836653	0.904837641	0.904838360	0.904837405
0.2	0.818730753	0.818729658	0.818729327	0.818730116	0.818730779	0.818729914
0.3	0.740818221	0.740817567	0.740817534	0.740817669	0.740817892	0.740817617
0.4	0.670320046	0.670320480	0.670320739	0.670320224	0.670319988	0.670320325
0.5	0.606530660	0.606531784	0.606532207	0.606531326	0.606530834	0.606531513
0.6	0.548811636	0.548812524	0.548812934	0.548812086	0.548811627	0.548812264
0.7	0.496585304	0.496585344	0.496585583	0.496585121	0.496584944	0.496585206
0.8	0.449328964	0.449328427	0.449328428	0.449328497	0.449328692	0.449328457
0.9	0.406569660	0.406569434	0.406569283	0.406569672	0.406570054	0.406569563
1	0.367879441	0.367879441	0.367879441	0.367879441	0.367879441	0.367879441

Table 5. Solutions of example 2 in Galerkin method

x	Exact	Legendre-LCP	Chebyshev-CCP	Bernstein-JCP	Lagrange-ESCP	Fibonacci-GLCP
0.	1.000000000	1.000000000	1.000000000	1.000000000	1.000000000	1.000000000
0.1	1.105170918	1.105170213	1.105169417	1.105171419	1.105173304	1.105170874
0.2	1.221402758	1.221400338	1.221399830	1.221401287	1.221402990	1.221400838
0.3	1.349858808	1.349857556	1.349857808	1.349857597	1.349858176	1.349857533
0.4	1.491824698	1.491826169	1.491827164	1.491825292	1.491824689	1.491825620
0.5	1.648721271	1.648724368	1.648725755	1.648723001	1.648721761	1.648723539
0.6	1.822118800	1.822121010	1.822122279	1.822119788	1.822118734	1.822120264
0.7	2.013752707	2.013752392	2.013753071	2.013751894	2.013751763	2.013752061
0.8	2.225540928	2.225539027	2.225538900	2.225539481	2.225540514	2.225539247
0.9	2.459603111	2.459602416	2.459601769	2.459603380	2.459604871	2.459602945
1	2.718281828	2.718281822	2.718281707	2.718281828	2.718281631	2.718281844