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# Rough Interior Ideals and Rough Quasi-Ideals in Approximation Spaces of Semigroups under Preorder and Compatible Relations

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**Abstract**

In this paper, a rough interior ideal and a rough quasi-ideal in an approximation space of a semigroup under a preorder and compatible relation are proposed. As introduced above, corresponding examples are presented. Next, we provide a sufficient condition for the rough interior ideal (resp., rough quasi-ideal). Finally, we give a necessary and sufficient condition for a homomorphic image of the rough interior ideal (resp., rough quasi-ideal).

**Keywords**: Rough interior ideal, Rough quasi-ideal, Approximation space, Semigroup, Preorder and compatible relation

## 1. Introduction

In 1982, Pawlak introduced Pawlak's rough set theory. Mathematically, this is a classical tool for assessing problems and decision problems in many fields with respect to set theory. Pawlak's rough set theory has been regarded as an approximation processing model of sets induced by equivalence relations. Based on a Pawlak's approximation space induced by an equivalence relation **(**a pair of a non-empty universal set with an equivalence relation**)**, a non-empty subset of the given universe is approximated by the following sets (Pawlak, 1982).

The Pawlak's upper approximation set is the union of equivalent classes which have a non-empty intersection with the given non-empty subset **(**The set of all possibly elements with respect to a property of the given non-empty subset**).**

The Pawlak's lower approximation set is the union of equivalent classes which are subsets of the given non-empty subset **(**The set of all certainly elements with respect to a property of the given non-empty subset**).**

The Pawlak's boundary region is a difference of the upper approximation and the lower approximation **(**The set of all uncertain elements with respect to a property of the given non-empty subset**).**

The Pawlak's rough set of the given non-empty subset is defined by meaning of a pair of upper and lower approximations, where the difference of upper and lower approximations is a non-empty set**.** Otherwise, the given non-empty subset is said to be a Pawlak's definable set**.**

The combination of semigroup theory and Pawlak's rough set theory is one of the most interesting varieties. In 1997, Kuroki proposed the concepts of rough semigroups (resp. ideals) in approximation spaces of semigroups induced by congruence relations, and provided sufficient conditions of rough semigroups (resp. ideals) (Kuroki, 1997). In 2006, Xiao and Zhang introduced the concepts of rough completely prime ideals in approximation spaces of semigroups induced by congruence relations, and provided sufficient conditions of rough completely prime ideals (Xiao & Zhang, 2006). They verified the relationship between rough completely prime ideals (resp. ideals) and the homomorphic image of rough completely prime ideals (resp. ideals) under homomorphism problems. In 2012, Yaqoob, Aslam, and Chinram proposed the notions of rough prime bi-ideals in approximation spaces of semigroups induced by congruence relations, and provided sufficient conditions of rough prime bi-ideals (Yaqoob, Aslam, & Chinram, 2012).

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Based on a binary relation between two universes, Prasertpong and Siripitukdet introduced a generalized rough set in 2019. Especially, they defined rough semigroups, rough ideals and rough completely prime ideals in semigroups under approximation spaces induced by preorder and compatible relations, including provided sufficient conditions for them, and proved necessary and sufficient conditions for their homomorphic images (Prasertpong & Siripitukdet, 2019).

In this research, after providing some fundamentals of semigroups, binary relations and generalized rough sets in Section 2, we firstly introduce rough interior ideals and rough quasi-ideals in semigroups under approximation spaces induced by preorder and compatible relations in Section 3. Then, we provide sufficient conditions for them. In the end, we give a necessary and sufficient condition for a homomorphic image of the rough interior ideal (resp., rough quasi-ideal).

## 2. Preliminaries

In this section, we recall important terms which will be used in a subsequent section.

### 2.1 Some basic attributes in semigroups

**Definition 2.1.1** A *semigroup* $(S,\cdot)$ is defined as an algebraic system, where $S$ is a non-empty set and " $\cdot$ " is an associative binary operation on $S$. Throughout this paper, $S$ stands for a semigroup (Clifford & Preston, 1961).

**Definition 2.1.2** An element $s$ in $S$ is called an *idempotent element* if $s^2 = s$. For any $X \subseteq S$, the set of all idempotents in $X$ is denoted by $E(X)$ (Clifford & Preston, 1961).

**Definition 2.1.3** Let $X$ be a non-empty subset of $S$. $X$ is called a *right ideal* (resp., a *left ideal*) of $S$ if $XS \subseteq X$ (resp., $SX \subseteq X$). $X$ is called an *ideal* of $S$ if it is a right ideal and a left ideal of $S$ (Clifford & Preston, 1961).

**Definition 2.1.4** $S$ is called a *semisimple semigroup* if $X = E(X)$ for every ideal $X$ of $S$ (Clifford & Preston, 1961).

**Definition 2.1.5** $S$ is called a *commutative semigroup* if $s_1s_2 = s_2s_1$ for all $s_1, s_2 \in S$ (Howie, 1976).

**Definition 2.1.6** A non-empty subset $X$ of $S$ is called an *interior ideal* of $S$ if $SXS \subseteq X$ (Mordeson, Malik, & Kuroki, 2010).

**Theorem 2.1.7** $S$ is a semisimple semigroup if and only if

$$X \cap Y = XY$$

for every interior ideal $X$ and every ideal $Y$ of $S$ (Mordeson, Malik, & Kuroki, 2010).

**Definition 2.1.8** A non-empty subset $X$ of $S$ is called a *quasi-ideal* of $S$ if $XS \cap SX \subseteq X$ (Mordeson, Malik, & Kuroki, 2010).

### 2.2 Some basic definitions of relations

Throughout this paper, $U$ and $V$ denote two non-empty universal sets.

**Definition 2.2.1** (Zach, 2017) Let $P(U \times V)$ be a collection of all subsets of $U \times V$. An element in $P(U \times V)$ is called a *binary relation from $U$ to $V$*. An element in $P(U \times V)$ is called a *binary relation on $U$* if $U = V$.

**Definition 2.2.2** (Zach, 2017)

(1) $\theta$ is called *reflexive* if $(u,u) \in \theta$ for all $u \in U$.

(2) $\theta$ is called *transitive* if for each $u_1, u_2, u_3 \in U$, $(u_1,u_2) \in \theta$ and $(u_2, u_3) \in \theta$ imply $(u_1,u_3) \in \theta$.

(3) $\theta$ is called *symmetric* if for each $u_1, u_2 \in U$, $(u_1,u_2) \in \theta$ implies $(u_2,u_1) \in \theta$.

(4) If $\theta$ is reflexive and transitive, then $\theta$ is called a *preorder relation*.

(5) If $\theta$ is reflexive, transitive and symmetric, then $\theta$ is called an *equivalence relation*.

**Definition 2.2.3** Let $\theta$ be an equivalence relation on $U$. For an element $u \in U$, the set (Zach, 2017)

$$[u]_\theta \coloneqq \{v \in V : (u,v) \in \theta\} \qquad (2.2.1)$$

is called an *equivalence class of $u$ induced by $\theta$*.

**Definition 2.2.4** Let $\theta$ be a binary relation on $S$. $\theta$ is called *compatible (with the operation on S)* if for each $s_1, s_2, s_3 \in S$, $(s_1, s_2) \in \theta$ implies $(s_1s_3, s_2s_3) \in \theta$ and $(s_3s_1, s_3s_2) \in \theta$. If $\theta$ is an equivalence relation with compatible, then it is called a congruence (Howie, 1976).

### 2.3 Fundamentals of generalized rough sets in semigroups

**Definition 2.3.1** Let $\theta$ be a binary relation from $U$ to $V$. For an element $u \in U$, the set

$$S_\theta(u) \coloneqq \{v \in V : (u,v) \in \theta\} \qquad (2.3.1)$$

is called a *successor class of $u$ induced by $\theta$*

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(Prasertpong & Siripitukdet, 2019).

**Definition 2.3.2** Let $\theta$ be a binary relation from $U$ to $V$. For an element $u_1 \in U$, the set in Equation (2.3.2) as
$PS_\theta(u_1) := \{u_2 \in U : S_\theta(u_2) \subseteq S_\theta(u_1)\}$
is called a *portion of the successor class of* $u_1$ *induced by* $\theta$. $\mathcal{PS}_\theta(U)$ is denoted as a collection of $PS_\theta(u)$ for all $u \in U$ (Prasertpong & Siripitukdet, 2019).

**Definition 2.3.3** Let $\theta$ be a binary relation from $U$ to $V$. The triple $(U, V, \mathcal{PS}_\theta(U))$ is called a $\mathcal{PS}_\theta(U)$-*approximation space*. If we change $V$ to $U$, then $(U, V, \mathcal{PS}_\theta(U))$ is replaced by a pair $(U, \mathcal{PS}_\theta(U))$ (Prasertpong & Siripitukdet, 2019).

**Definition 2.3.4** Let $(U, V, \mathcal{PS}_\theta(U))$ be a $\mathcal{PS}_\theta(U)$-approximation space and let $X$ be a non-empty subset of $U$. The set in Equation (2.3.3) as

$$\overline{\theta}(X) := \{u \in U : PS_\theta(u) \cap X \neq \emptyset\}$$

is called a $\mathcal{PS}_\theta(U)$-*upper approximation of* $X$ (The set of all possibly elements with respect to a property of the given non-empty subset). The set in Equation (2.3.4) as

$$\underline{\theta}(X) := \{u \in U : PS_\theta(u) \subseteq X\}$$

is called a $\mathcal{PS}_\theta(U)$-*lower approximation of* $X$ (The set of all certainly elements with respect to a property of the given non-empty subset). The set in Equation (2.3.5) as

$$\theta_{bnd}(X) := \overline{\theta}(X) - \underline{\theta}(X)$$

is called a $\mathcal{PS}_\theta(U)$-*boundary region of* $X$ (The set of all uncertain elements with respect to a property of the given non-empty subset). If $\theta_{bnd}(X) \neq \emptyset$, then $\theta(X) := (\overline{\theta}(X), \underline{\theta}(X))$ is called a $\mathcal{PS}_\theta(U)$-*rough set of* $X$. In this way, we say that $X$ is a $\mathcal{PS}_\theta(U)$-*rough set*. If $\theta_{bnd}(X) = \emptyset$, then $X$ is called a $\mathcal{PS}_\theta(U)$-*definable set* (Prasertpong & Siripitukdet, 2019).

**Remark 2.3.5** Based on Definition 2.3.4, note that every Pawlak's rough set is a $\mathcal{PS}_\theta(U)$-rough set. Conversely, it is not true in general. Indeed, a $\mathcal{PS}_\theta(U)$-rough set is a generalization of a Pawlak's rough set whenever a binary relation is an equivalence relation, that is, Equation (2.2.1) and Equation (2.3.2) are identical. As proposed above, a corresponding example is considered as Example 1 in Prasertpong and Siripitukdet (2019).

**Proposition2.3.6** Let $(U, V, \mathcal{PS}_\theta(U))$ be a $\mathcal{PS}_\theta(U)$-approximation space. If $X$ and $Y$ are non-empty subsets of $U$, then the following statements hold.

(1) $\overline{\theta}(U) = U$ and $\underline{\theta}(U) = U$.

(2) $X \subseteq \overline{\theta}(X)$ and $\underline{\theta}(X) \subseteq X$.

(3) $\underline{\theta}(X \cap Y) = \underline{\theta}(X) \cap \underline{\theta}(Y)$.

(4) If $X \subseteq Y$, then $\overline{\theta}(X) \subseteq \overline{\theta}(Y)$ and $\underline{\theta}(X) \subseteq \underline{\theta}(Y)$ (Prasertpong & Siripitukdet, 2019).

**Definition 2.3.7** Let $(U, V, \mathcal{PS}_\theta(U))$ be a $\mathcal{PS}_\theta(U)$-approximation space and let $X$ be a non-empty subset of $U$. If $\underline{\theta}(U)$ is a non-empty proper subset of $X$, then $X$ is called a *set over a non-empty interior set* (Prasertpong & Siripitukdet, 2019).

**Remark 2.3.8** Using the similar method in the proof of Proposition 4 in Prasertpong and Siripitukdet (2019), it is easy to see that if $X$ is a non-empty subset of $U$ over a non-empty interior set, then it is a $\mathcal{PS}_\theta(U)$-rough set.

**Definition 2.3.9** Let $(S, \mathcal{PS}_\theta(S))$ be a $\mathcal{PS}_\theta(S)$-approximation space. $(S, \mathcal{PS}_\theta(S))$ is called a $\mathcal{PS}_\theta(S)$-*approximation space type* $PCR$ if $\theta$ is a preorder and compatible relation (Prasertpong & Siripitukdet, 2019).

**Definition 2.3.10** Let $(S, \mathcal{PS}_\theta(S))$ be a $\mathcal{PS}_\theta(S)$-approximation space type $PCR$. $\theta$ is called a *complete relation* if

$$(PS_\theta(s_1))(PS_\theta(s_2)) = PS_\theta(s_1 s_2)$$

for all $s_1, s_2 \in S$. $(S, \mathcal{PS}_\theta(S))$ is called a $\mathcal{PS}_\theta(S)$-*approximation space type* $CR$ if $\theta$ is a complete relation (Prasertpong & Siripitukdet, 2019).

**Remark 2.3.11** According to Definitions 2.3.9 and 2.3.10, every $\mathcal{PS}_\theta(S)$-approximation space type $CR$ is a $\mathcal{PS}_\theta(S)$-approximation space type $PCR$.

**Proposition 2.3.12** Let $(S, \mathcal{PS}_\theta(S))$ be a $\mathcal{PS}_\theta(S)$-approximation space. Then the following statements hold.

(1) If $(S, \mathcal{PS}_\theta(S))$ is a $\mathcal{PS}_\theta(S)$-approximation space type $PCR$, then $(\overline{\theta}(X))(\overline{\theta}(Y)) \subseteq \overline{\theta}(XY)$ for every non-empty subsets $X$ and $Y$ of $S$.

(2) If $(S, \mathcal{PS}_\theta(S))$ is a $\mathcal{PS}_\theta(S)$-approximation space type $CR$, then $(\underline{\theta}(X))(\underline{\theta}(Y)) \subseteq \underline{\theta}(XY)$ for every non-empty subsets $X$ and $Y$ of $S$ (Prasertpong & Siripitukdet, 2019).

**Theorem 2.3.13** Let $(S, \mathcal{PS}_\theta(S))$ be a $\mathcal{PS}_\theta(S)$-approximation space and let $X$ be a non-empty subset of $S$. If $X$ is an ideal of $S$ in $(S, \mathcal{PS}_\theta(S))$ type $PCR$, then $\overline{\theta}(X)$ is an ideal of $S$ (Prasertpong & Siripitukdet, 2019).

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**Proposition 2.3.14** Let $f$ be an epimorphism from $S$ in $(S, \mathcal{PS}_\theta(S))$ to $T$ in $\left(T, \mathcal{PS}_\vartheta(T)\right)$, where the binary relation $\theta$ is defined by Equation (2.3.6) as

$$\theta = \{(s_1, s_2) \in S \times S : (f(s_1), f(s_2)) \in \vartheta\}.$$

Then the following statements hold.

(1) $f(\overline{\theta}(X)) = \overline{\vartheta}(f(X))$ for every non-empty subset $X$ of $S$.

(2) If $f$ is injective, then $f(\underline{\theta}(X)) = \underline{\vartheta}(f(X))$ for every non-empty subset $X$ of $S$ (Prasertpong & Siripitukdet, 2019).

## 3. Main Results

In this section, we introduce a rough interior ideal and a rough quasi-ideal in a $\mathcal{PS}_\theta(S)$-approximation space type $PCR$. Then we provide sufficient conditions for them. Based on homomorphism problem in semigroup, we give a necessary and sufficient condition for a homomorphic image of the rough interior ideal (resp., rough quasi-ideal).

**Definition 3.1** Let $\left(S, \mathcal{PS}_\theta(S)\right)$ be a $\mathcal{PS}_\theta(S)$-approximation space type $PCR$ and let $X$ be a non-empty subset of $S$. $X$ is called a $\mathcal{PS}_\theta(S)$-*upper rough interior ideal* if $\overline{\theta}(X)$ is an interior ideal of $S$. $X$ is called a $\mathcal{PS}_\theta(S)$-*lower rough interior ideal* if $\underline{\theta}(X)$ is an interior ideal of $S$. $X$ is called a $\mathcal{PS}_\theta(S)$-*rough interior ideal* if it is a $\mathcal{PS}_\theta(S)$-upper rough interior ideal, a $\mathcal{PS}_\theta(S)$-lower rough interior ideal and a $\mathcal{PS}_\theta(S)$-rough set. Similarly, we can define a $\mathcal{PS}_\theta(S)$-rough quasi-ideal.

We consider the following example.

**Example 3.2** Based on Example 3 in Prasertpong and Siripitukdet (2019), we let $S = \{s_1, s_2, s_3, s_4, s_5\}$ be a semigroup with multiplication rules defined by the following table.

**Table 1.** The multiplication table on $S$.

| | $s_1$ | $s_2$ | $s_3$ | $s_4$ | $s_5$ |
|---|---|---|---|---|---|
| $s_1$ | $s_1$ | $s_1$ | $s_3$ | $s_1$ | $s_5$ |
| $s_2$ | $s_1$ | $s_2$ | $s_3$ | $s_1$ | $s_5$ |
| $s_3$ | $s_3$ | $s_3$ | $s_3$ | $s_3$ | $s_3$ |
| $s_4$ | $s_1$ | $s_1$ | $s_3$ | $s_4$ | $s_5$ |
| $s_5$ | $s_5$ | $s_5$ | $s_3$ | $s_5$ | $s_5$ |

Let $\theta$ be a relation defined as follows:
$\{(s_1, s_1), (s_1, s_2), (s_1, s_4), (s_2, s_1), (s_2, s_2), (s_2, s_4), (s_3, s_3), (s_3, s_5), (s_4, s_1), (s_4, s_2), (s_4, s_4), (s_5, s_5)\}$.
Then, it is easily seen that $\theta$ is a preorder and compatible relation. According to Equation (2.3.1) in Definition 2.3.1, it follows that

$$S_\theta(s_1) = \{s_1, s_2, s_4\},$$
$$S_\theta(s_2) = \{s_1, s_2, s_4\},$$
$$S_\theta(s_3) = \{s_3, s_5\},$$
$$S_\theta(s_4) = \{s_1, s_2, s_4\},$$
$$S_\theta(s_5) = \{s_5\}.$$

According to Equation (2.3.2) in Definition 2.3.2, it follows that

$$PS_\theta(s_1) = S_\theta(s_1),$$
$$PS_\theta(s_2) = S_\theta(s_2),$$
$$PS_\theta(s_3) = S_\theta(s_3),$$
$$PS_\theta(s_4) = S_\theta(s_4),$$
$$PS_\theta(s_5) = S_\theta(s_5).$$

Suppose that $X = \{s_2, s_3, s_5\}$ is a non-empty subset of $S$, which is a set for an approximation in $\left(S, \mathcal{PS}_\theta(S)\right)$ type $PCR$. Then, by Equations (2.3.3) and (2.3.4) in Definition 2.3.4, we see that $\overline{\theta}(X) = S$ and $\underline{\theta}(X) = \{s_3, s_5\}$, respectively. Hence $\theta_{bnd}(X) \neq \emptyset$, and so $X$ is a $\mathcal{PS}_\theta(S)$-rough set. Moreover, it is easily verified that $\overline{\theta}(X)$ and $\underline{\theta}(X)$ are interior ideals and quasi-ideals. Moreover, we observe that $X$ is a $\mathcal{PS}_\theta(S)$-(resp., upper, lower) rough interior ideal and $X$ is a $\mathcal{PS}_\theta(S)$-(resp., upper, lower) rough quasi-ideal.

We now come to main results.

**Theorem 3.3** Let $\left(S, \mathcal{PS}_\theta(S)\right)$ be a $\mathcal{PS}_\theta(S)$-approximation space and let $X$ be a non-empty subset of $S$. Then we have the following statements.

(1) If $X$ is an interior ideal of $S$ in $\left(S, \mathcal{PS}_\theta(S)\right)$ type $PCR$, then it is a $\mathcal{PS}_\theta(S)$-upper rough interior ideal.

(2) If $X$ is an interior ideal of $S$ in $\left(S, \mathcal{PS}_\theta(S)\right)$ type $CR$ with respect to a non-empty $\underline{\theta}(X)$, then it is a $\mathcal{PS}_\theta(S)$-lower rough interior ideal.

(3) If $X$ is an interior ideal of $S$ over a non-empty interior set in $\left(S, \mathcal{PS}_\theta(S)\right)$ type $CR$, then it is a $\mathcal{PS}_\theta(S)$-rough interior ideal.

**Proof.** (1) Suppose that $X$ is an interior ideal of $S$ in $\left(S, \mathcal{PS}_\theta(S)\right)$ type $PCR$. Then $SXS \subseteq X$. By Proposition 2.3.6 (2), we get $X \subseteq \overline{\theta}(X)$. Thus $\overline{\theta}(X) \neq \emptyset$. By Proposition 2.3.6 (4), we obtain $\overline{\theta}(SXS) \subseteq \overline{\theta}(X)$. By Proposition 2.3.6 (1), we have $\overline{\theta}(S) = S$. From Proposition 2.3.12 (1), it follows that

$$S\left(\overline{\theta}(X)\right)S = \left(\overline{\theta}(S)\right)\left(\overline{\theta}(X)\right)\left(\overline{\theta}(S)\right) \subseteq \overline{\theta}(SXS) \subseteq \overline{\theta}(X).$$

Hence $\overline{\theta}(X)$ is an interior ideal of $S$. Therefore, $X$ is a $\mathcal{PS}_\theta(S)$-upper rough interior ideal.

(2) From Propositions 2.3.6 (1) and (4), 2.3.12 (2) and using the similar method in the proof of argument (1), we can prove that the statement is true under $\left(S, \mathcal{PS}_\theta(S)\right)$ type $CR$.

(3) From Remark 2.3.8 and arguments (1) and

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(2), we can prove that the statement is true.

**Theorem 3.4** Let $(S, \mathcal{PS}_\theta(S))$ be a $\mathcal{PS}_\theta(S)$-approximation space and let $X$ be a non-empty subset of $S$. Then we have the following statements.

(1) If $X$ is a quasi-ideal of $S$ in $(S, \mathcal{PS}_\theta(S))$ type $PCR$, where $S$ is semisimple and commutative, then it is a $\mathcal{PS}_\theta(S)$-upper rough quasi-ideal.

(2) If $X$ is a quasi-ideal of $S$ in $(S, \mathcal{PS}_\theta(S))$ type $CR$ with respect to a non-empty $\underline{\theta}(X)$, then it is a $\mathcal{PS}_\theta(S)$-lower rough quasi-ideal.

(3) If $X$ is a quasi-ideal of $S$ over a non-empty interior set in $(S, \mathcal{PS}_\theta(S))$ type $CR$, where $S$ is semisimple and commutative, then it is a $\mathcal{PS}_\theta(S)$-rough quasi-ideal.

**Proof.** (1) Suppose that $X$ is a quasi-ideal of $S$ in $(S, \mathcal{PS}_\theta(S))$ type $PCR$, where $S$ is semisimple and commutative. Then $XS \cap SX \subseteq X$. By Proposition 2.3.6 (4), we get that $\overline{\theta}(XS \cap SX) \subseteq \overline{\theta}(X)$. Note that $XS$ is a right ideal and $SX$ is a left ideal of $S$. Since $S$ is commutative, $XS$ and $SX$ are ideals of $S$. Then, by Theorem 2.3.13, we obtain that $\overline{\theta}(XS)$ and $\overline{\theta}(SX)$ are ideals of $S$. Note that $XS$ is an interior ideal of $S$. Then, by Theorem 3.3 (1), we obtain $\overline{\theta}(XS)$ is an interior ideal of $S$. Since $S$ is semisimple, by Theorem 2.1.7, we get that

$$XS \cap SX = (XS)(SX)$$

and

$$\overline{\theta}(XS) \cap \overline{\theta}(SX) = \left(\overline{\theta}(XS)\right)\left(\overline{\theta}(SX)\right).$$

From Propositions 2.3.6 (1) and 2.3.12 (1), it follows that

$$\begin{aligned}
&\left(\overline{\theta}(X)\right) S \cap S\left(\overline{\theta}(X)\right) \\
&= \left(\overline{\theta}(X)\right)\left(\overline{\theta}(S)\right) \cap \left(\overline{\theta}(S)\right)\left(\overline{\theta}(X)\right) \\
&\subseteq \overline{\theta}(XS) \cap \overline{\theta}(SX) \\
&= \left(\overline{\theta}(XS)\right)\left(\overline{\theta}(SX)\right) \\
&\subseteq \overline{\theta}\left((XS)(SX)\right) \\
&= \overline{\theta}\left((XS) \cap (SX)\right) \\
&\subseteq \overline{\theta}(X).
\end{aligned}$$

Thus $\overline{\theta}(X)$ is a quasi-ideal of $S$. This means that $X$ is a $\mathcal{PS}_\theta(S)$-upper rough quasi-ideal.

(2) Suppose that $X$ is a quasi-ideal of $S$ in $(S, \mathcal{PS}_\theta(S))$ type $CR$ with respect to a non-empty $\underline{\theta}(X)$. Then $XS \cap SX \subseteq X$. From Proposition 2.3.6 (1), (3) and (4) and Proposition 2.3.12 (2), it follows that

$$\begin{aligned}
&\left(\underline{\theta}(X)\right) S \cap S\left(\underline{\theta}(X)\right) \\
&= \left(\underline{\theta}(X)\right)\left(\underline{\theta}(S)\right) \cap \left(\underline{\theta}(S)\right)\left(\underline{\theta}(X)\right) \\
&\subseteq \underline{\theta}(XS) \cap \underline{\theta}(SX) \\
&= \underline{\theta}\left((XS) \cap (SX)\right) \\
&\subseteq \underline{\theta}(X).
\end{aligned}$$

Hence $\underline{\theta}(X)$ is a quasi-ideal of $S$. It follows that $X$ is a $\mathcal{PS}_\theta(S)$-lower rough quasi-ideal.

(3) From Remark 2.3.8 and arguments (1) and (2), we can prove that the statement is true.

**Theorem 3.5** Let $f$ be an epimorphism from $S$ in $(S, \mathcal{PS}_\theta(S))$ to $T$ in $(T, \mathcal{PS}_\vartheta(T))$ type $PCR$, where $\theta$ is defined as Equation (2.3.6) in Proposition 2.3.14. If $X$ is a non-empty subset of $S$ and $f$ is injective, then we have the following statements.

(1) $f(X)$ is a $\mathcal{PS}_\vartheta(T)$-upper rough interior ideal if and only if $X$ is a $\mathcal{PS}_\theta(S)$-upper rough interior ideal.

(2) $f(X)$ is a $\mathcal{PS}_\vartheta(T)$-lower rough interior ideal if and only if $X$ is a $\mathcal{PS}_\theta(S)$-lower rough interior ideal.

(3) $f(X)$ is a $\mathcal{PS}_\vartheta(T)$-rough interior ideal if and only if $X$ is a $\mathcal{PS}_\theta(S)$-rough interior ideal.

**Proof.** (1) Suppose that $f(X)$ is a $\mathcal{PS}_\vartheta(T)$-upper rough interior ideal. Then, we have that

$$T\left(\overline{\vartheta}\left(f(X)\right)\right)T \subseteq \overline{\vartheta}\left(f(X)\right).$$

Let $s_1 \in S\left(\overline{\theta}(X)\right)S$. By Proposition 2.3.14 (1), we obtain

$$\begin{aligned}
&f(s_1) \in f\left(S\left(\overline{\theta}(X)\right)S\right) = T\left(\overline{\vartheta}\left(f(X)\right)\right)T \\
&\subseteq \overline{\vartheta}\left(f(X)\right) = f\left(\overline{\theta}(X)\right).
\end{aligned}$$

Thus, there exists $s_2 \in \overline{\theta}(X)$ such that $f(s_1) = f(s_2)$. Hence $PS_\theta(s_2) \cap X \neq \emptyset$. Since $f$ is injective, we have $s_1 = s_2$. Thus $PS_\theta(s_1) \cap X \neq \emptyset$. Hence $s_1 \in \overline{\theta}(X)$. Whence $S\left(\overline{\theta}(X)\right)S \subseteq \overline{\theta}(X)$. Hence $\overline{\theta}(X)$ is an interior ideal of $S$. Therefore, $X$ is a $\mathcal{PS}_\theta(S)$-upper rough interior ideal.

Conversely, assume that $X$ is a $\mathcal{PS}_\theta(S)$-upper rough interior ideal. Then $S\left(\overline{\theta}(X)\right)S \subseteq \overline{\theta}(X)$. Thus $f\left(S\left(\overline{\theta}(X)\right)S\right) \subseteq f\left(\overline{\theta}(X)\right)$. By Proposition 2.3.14 (1), we see that

$$\begin{aligned}
&T\left(\overline{\vartheta}\left(f(X)\right)\right)T = f\left(S\left(\overline{\theta}(X)\right)S\right) \\
&\subseteq f\left(\overline{\theta}(X)\right) = \overline{\vartheta}\left(f(X)\right).
\end{aligned}$$

Hence $\overline{\vartheta}\left(f(X)\right)$ is an interior ideal of $T$. Therefore $f(X)$ is a $\mathcal{PS}_\vartheta(T)$-upper rough interior ideal.

(2) By Proposition 2.3.14 (2) and using the similar method in the proof of argument (1), we can prove that the statement is true.

(3) Under the injective mapping $f$, the proof is obvious from arguments (1) and (2).

**Theorem 3.6** Let $f$ be an epimorphism from $S$ in $(S, \mathcal{PS}_\theta(S))$ to $T$ in $(T, \mathcal{PS}_\vartheta(T))$ type $PCR$, where $\theta$ is defined as Equation (2.3.6) in Proposition 2.3.14. If $X$ is a non-empty subset of $S$ and $f$ is injective, then we have the following statements.

(1) $f(X)$ is a $\mathcal{PS}_\vartheta(T)$-upper rough quasi-ideal if and only if $X$ is a $\mathcal{PS}_\theta(S)$-upper rough quasi-ideal.

(2) $f(X)$ is a $\mathcal{PS}_\vartheta(T)$-lower rough quasi-ideal if and only if $X$ is a $\mathcal{PS}_\theta(S)$-lower rough quasi-ideal.

(3) $f(X)$ is a $\mathcal{PS}_\vartheta(T)$-rough quasi-ideal if and only if $X$ is a $\mathcal{PS}_\theta(S)$-rough quasi-ideal.

**Proof.** (1) Suppose that $f(X)$ is a $\mathcal{PS}_\vartheta(T)$-upper rough quasi-ideal. Then $\overline{\vartheta}\left(f(X)\right)$ is a quasi-ideal of $T$. Thus $\left(\overline{\vartheta}\left(f(X)\right)\right)T \cap T\left(\overline{\vartheta}\left(f(X)\right)\right) \subseteq \overline{\vartheta}\left(f(X)\right)$. Let $s_1 \in \left(\overline{\theta}(X)\right)S \cap S\left(\overline{\theta}(X)\right)$. Then, we have that $s_1 \in \left(\overline{\theta}(X)\right)S$ and $s_1 \in S\left(\overline{\theta}(X)\right)$. Hence $f(s_1) \in f\left(\left(\overline{\theta}(X)\right)S\right)$ and $f(s_1) \in f\left(S\left(\overline{\theta}(X)\right)\right)$. By

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Proposition 2.3.14 (1), we get that $f(s_1) \in \left(\overline{\vartheta}\left(f(X)\right)\right)T \cap T\left(\overline{\vartheta}\left(f(X)\right)\right)$. We observe that

$$f(s_1) \in \left(\overline{\vartheta}\left(f(X)\right)\right)T \cap T\left(\overline{\vartheta}\left(f(X)\right)\right) \subseteq \overline{\vartheta}\left(f(X)\right) = f\left(\overline{\theta}(X)\right).$$

Thus, there exists $s_2 \in \overline{\theta}(X)$ such that $f(s_1) = f(s_2)$. Hence $PS_\theta(s_2) \cap X \neq \emptyset$. Since $f$ is injective, we have $s_1 = s_2$. Thus $PS_\theta(s_1) \cap X \neq \emptyset$. Hence $s_1 \in \overline{\theta}(X)$. Whence $\left(\overline{\theta}(X)\right)S \cap S\left(\overline{\theta}(X)\right) \subseteq \overline{\theta}(X)$, which yields $\overline{\theta}(X)$ is a quasi-ideal of $S$. It follows that $X$ is a $\mathcal{PS}_\theta(S)$-upper rough quasi-ideal.

Conversely, suppose $X$ is a $\mathcal{PS}_\theta(S)$-upper rough quasi-ideal. Then $\overline{\theta}(X)$ is a quasi-ideal of $S$. Thus

$$\left(\overline{\theta}(X)\right)S \cap S\left(\overline{\theta}(X)\right) \subseteq \overline{\theta}(X).$$

Hence

$$f\left(\left(\overline{\theta}(X)\right)S \cap S\left(\overline{\theta}(X)\right)\right) \subseteq f\left(\overline{\theta}(X)\right).$$

Since $f$ is injective, it is easy to prove that

$$f\left(\left(\overline{\theta}(X)\right)S\right) \cap f\left(S\left(\overline{\theta}(X)\right)\right) = f\left(\left(\overline{\theta}(X)\right)S \cap S\left(\overline{\theta}(X)\right)\right).$$

By Proposition 2.3.14 (1), we obtain that

$$\left(\overline{\vartheta}\left(f(X)\right)\right)T \cap T\left(\overline{\vartheta}\left(f(X)\right)\right) = f\left(\left(\overline{\theta}(X)\right)S\right) \cap f\left(S\left(\overline{\theta}(X)\right)\right) \subseteq f\left(\overline{\theta}(X)\right) = \overline{\vartheta}\left(f(X)\right).$$

Thus $\overline{\vartheta}\left(f(X)\right)$ is a quasi-ideal of $T$. Consequently $f(X)$ is a $\mathcal{PS}_\vartheta(T)$-upper rough quasi-ideal.

(2) By Proposition 2.3.14 (2) and using the similar method in the proof of argument (1), we can prove that the statement is true.

(3) Under the injective mapping $f$, the proof is obvious from arguments (1) and (2).

## 4. Conclusions and Suggestions

Based on the generalized rough set model in Prasertpong and Siripitukdet **(**2019**)**, we introduced a rough interior ideal and a rough quasi**-**ideal in a semigroup under an approximation space induced by a preorder and compatible relation and derived sufficient conditions for them**.** Also, we proved a relationships between the interior ideal **(**resp**.** quasi**-**ideal**)** and its homomorphic image**.** Observe that we obtained results in semigroups by using a non**-**symmetric relation, which differ from Kuroki, **(**1997**)**, Xiao and Zhang **(**2006**)** and Yaqoob et al. (2012**).** Then, the novel rough set in Prasertpong and Siripitukdet **(**2019**)** can be used in a semigroup for approximation processings in terms of crisp sets as Section 3**.** In the end, we hope that main results in this work may provide a powerful tool for assessment and decision problems in various fields with respect to information sciences and computer sciences**.**

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