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A New Type of Extended Soft Set Operation: Complementary Extended Union Operation

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Abstract

Soft set theory was proposed by Molodtsov in 1999 to model some problems involving uncertainty. It has a wide range of theoretical and practical applications. Soft set operations constitute the basic building blocks of soft set theory. Many kinds of soft set operations have been described and applied in various ways since the inception of the theory. In this paper, to contribute to the theory, a new soft set operation, called complementary extended union operation, is defined, its properties are discussed in detail to obtain the relationship of each operation with other soft set operations, and the distributions of these operations over other soft set operations are examined. We obtain that the complementary extended union operation along with other certain types of soft set operations construct some well-known algebraic structure such as Boolean Algebra, De Morgan Algebra, semiring, and hemiring in the set of soft sets with a fixed parameter set. Since Boolean Algebra is fundamental in digital logic design, computer science, information retrieval, set theory and probability; De Morgan Algebra in logic and set theory, computer science, artificial intelligence, circuit design; semirings in theoretical computer science, optimization problems, economics, cryptography and coding theory, and hemirings in combinatorics, mathematical economics, theoretical computer science, these algebraic structures provide essential tools for various applications, facilitating the analysis, design, and optimization of systems across many disciplines, and thus this study is expected to contribute to decision-making methods and cryptography based on soft sets.

Keywords: Soft set, Complementary extended soft set operations, Semiring, Hemiring, Algebras

1. Introduction

Many theories have been put forward to explain uncertainties for years. One of the most important theories in this field is the theory of fuzzy sets, proposed by Zadeh in 1965. A fuzzy set is defined through its membership function. The degree of membership is the real number between [0,1]. 0 indicates no membership, and 1 indicates full membership. However, due to the structural problems of the membership function, Molodtsov (1999) proposed soft set theory. The soft set theory eliminated the structural problems of the membership function. Molodtsov successfully applied soft set theory to continuous differentiable functions, operation research, Riemann integration, and many other fields. Soft set operations constitute the basis of soft set theory, and studies on both soft algebraic structures and soft decision-making methods are based on soft set operations. In this regard, Maji, Biswas, and Roy (2003) started inspiring studies on soft set operations. A more widely accepted definition of soft subset than the one defined by Maji et al. (2003) was given by Pei and Miao (2005).

When the studies of soft set operations such as Maji et al. (2003), Ali et al. (2009), Feng et al. (2010), Jiang et al. (2010), Ali et al. (2011), Fu (2011), Ge and Yang (2011), Neog and Sut (2011), Sezgin and Atagün (2011), Park et al. (2012), Singh and Onyeozili (2012), Zhu and Wen (2013), Onyeozili and Gwary (2014), Sen (2014), Husain and Shivani (2018), Sezgin et al. (2019), and Stojanovic (2021) are examined, it is seen that soft set operations proceed under two separate headings: restricted and extended operations. Moreover, Eren and Çalışıcı (2019) defined a new form of soft set operation for the literature, called the soft binary piecewise difference operation, Sezgin and Çalışıcı (2024) improved the work of Eren and Çalışıcı (2019) and studied the properties of the soft binary piecewise difference operation by comparing it with the difference operation in classical sets.

Çağman (2021) and Sezgin et al. (2023c) introduced new binary set operations, and these operations were transferred to soft sets as new restricted soft set operations and extended soft set operations by Aybek (2024). Besides, some new form of soft set operations, different from the restricted and extended forms of operations, were introduced by various authors (Akbulut, 2024; Demirci, 2024; Sarıalioğlu, 2024; Sezgin & Atagün, 2023; Sezgin & Aybek, 2023; Sezgin et al., 2023a; Sezgin et al., 2023b; Sezgin & Çağman, 2024; Sezgin & Dagtoros, 2023; Sezgin & Demirci, 2023; Sezgin & Sarıalioğlu, 2024; Sezgin & Yavuz, 2023a), and soft set operations, one of the most fundamental elements of soft set theory, has been studied by researchers since the theory was introduced. For other applications of soft sets as regards algebraic structures, we refer to: Çağman et al. (2012), Sezer (2014), Sezer et al. (2015), Muştuoğlu et al. (2016), Sezgin (2016), Tunçay and Sezgin (2016), Sezgin et al. (2017), Atagün and Sezgin (2018), Mahmood et al. (2018), Sezgin (2018), Jana et al. (2019), Özlü and Sezgin (2020), and Sezgin et al. (2022).

Moreover, different types of soft eqaulities were defined and some important equivalance relations were obtained with these different types of soft equalites as Jun and Yang (2011), Liu et al. (2012), Feng and Li (2013), Abbas et al. (2014), Abbas et al. (2017), Al-shami (2019), Al-shami and El-Shafei (2020). Studying the soft algebraic structures of an algebraic structure and other types of soft sets has been of interest by the researchers as Ali et al. (2015), Iftikhar and Mahmood (2018), and Mahmood (2020). For possible applications of graphs and network research concerning soft sets, we refer to Pant et al. (2024).

In the scope of algebra, one of the most important mathematical issues is analyzing the properties of the operation defined on a set to classify algebraic structures. Two primary categories of soft set collections are investigated within the context of soft sets as algebraic structures: The first represents a class of soft sets with a fixed set of parameters, whereas the second represents a class of soft sets with changing parameter sets. Depending on the extra properties, these two types of collections have different algebraic properties.

Boolean Algebras, De Morgan Algebras, semirings and hemirings provide essential tools for various applications, facilitating the analysis, design, and optimization of systems across many disciplines. Boolean algebra is fundamental in designing and simplifying digital circuits, algorithm design, data structure optimization, and coding theory, information retrieval, set theory and probability. De Morgan Algebra is essential in logic and set theory, especially in formal logic, helping to simplify complex logical expressions and set operations, computer science, design and optimization of algorithms, artificial intelligence, simplification of logic circuits by transforming expressions to their simplest forms. Semirings are crucial in theoretical computer science, automata theory, formal languages, and the analysis of algorithms, optimization problems, dynamic programming and shortest path algorithms, such as those used in network routing, mathematical modeling of economic systems, including cost-benefit analysis and resource allocation, constructing cryptographic protocols and coding theory. Hemiring are of importance as they generalize rings and semirings, providing a framework for more complex algebraic structures. They are used in the study of combinatorial structures and their properties, economic systems where subtraction may not always be feasible or meaningful and theoretical computer science, especially in automata theory.

In this study, we propose a new type of soft set operation called the complementary extended union operation and thoroughly discuss its properties to contribute to the theory of soft sets. To determine the relationship between the operation and other soft set operations, the distribution of complementary extended union operations over other kinds of soft set operations is examined. It is demonstrated that in the set of soft sets with a fixed parameter, the complementary extended union operation forms many well-known and significant algebraic structures in classical algebra, including semiring, hemiring, Boolean Algebra, and De Morgan Algebra, along with other specific kinds of soft set operations.

2. Preliminaries

Definition 2.1. Let U be the universal set, E be the parameter set, $\mathrm{P(U)}$ be the power set of U, and $\mathrm{V} \subseteq \mathrm{E}$. A pair (F, V) is called a soft set over U where F is a set-valued function such that $\mathrm{F}: \mathrm{V} \rightarrow \mathrm{P(U)}$ (Molodtsov, 1999).

Throughout this paper, the set of all the soft sets over U (no matter what the parameter set is) is designated by $\mathrm{S_E(U)}$. Let V be a fixed subset of E. Then, $\mathrm{S_V(U)}$ represents the collection of all soft sets over U with the fixed parameter set V.

Definition 2.2. $(G,\aleph)$ is called a relative null soft set (with respect to the parameter set $\aleph$), denoted by $\emptyset_\aleph$, if $G(t)=\emptyset$ for all $t\in\aleph$ and $(G,\aleph)$ is called a relative whole soft set (with respect to the parameter set $\aleph$), denoted by $U_\aleph$ if $G(t)=U$ for all $t\in\aleph$. The relative whole soft set U_E with respect to the universe set of parameters E is called the absolute soft set over U (Ali et al. 2009).

Definition 2.3. For two soft sets $(F,\aleph)$ and (G,R), we say that $(F,\aleph)$ is a soft subset of (G,R) and it is denoted by $(F,\aleph)\,\tilde{\subseteq}\,(G,R)$, if $\aleph\subseteq R$ and $F(t)\subseteq G(t)$, for all $t\in\aleph$. Two soft sets $(F,\aleph)$ and (G,R) are said to be soft equal if $(F,\aleph)$ is a soft subset of (G,R) and (G,R) is a soft subset of $(F,\aleph)$ (Pei & Miao, 2005).

Definition 2.4. The relative complement of a soft set $(F,\aleph)$, denoted by $(F,\aleph)^r$, is defined by $(F,\aleph)^r=(F^r,\aleph)$, where $F^r:\aleph\to P(U)$ is a mapping given by $(F,\aleph)^r=U\backslash F(t)$ for all $t\in\aleph$ (Ali et al. 2009). From now on, $U\backslash F(t)=[F(t)]'$ will be designated by $F'(t)$ for the sake of designation.

Çağman (2021) defined two new complements as inclusive and exclusive complements. Let + and θ denote inclusive and exclusive complements, respectively, and let V and Y be two sets. Then, these binary operations are as follows: $V+Y=V'\cup Y$, $V\theta Y=V'\cap Y'$. Sezgin et al. (2023c) analyzed the relations between these two operations and also defined three new binary operations and examined their relations with each other. Let V and Y be two sets $V*Y=V'\cup Y'$, $V\gamma Y=V'\cap Y$, $V\lambda Y=V\cup Y'$ and $\circledcirc$ denote the operations on sets. Then, restricted operations on soft sets, extended operations, extended operations with complement, soft binary piecewise operations, and soft binary piecewise operations with complement can be given in general form with the following generalized definitions:

Definition 2.5. Let $(F,V),(G,Y)\in S_E(U)$. The restricted $\circledcirc$ operation of (F,V) and (G,Y) is the soft set $(H,\mathbb{Z})$, denoted to be $(F,V)\circledcirc_{\Re}(G,Y)=(H,\mathbb{Z})$, where $\mathbb{Z}=V\cap Y\neq\emptyset$ and for all $\aleph\in\mathbb{Z}$, $H(\aleph)=F(\aleph)\circledcirc G(\aleph)$. Here, if $\mathbb{Z}=V\cap Y=\emptyset$, then $(F,V)\circledcirc(G,Y)=\emptyset_\emptyset$ (Ali et al., 2009; Aybek, 2024; Sezgin & Atagün, 2011).

Definition 2.6. Let $(F,V),(G,Y)\in S_E(U)$. The extended $\circledcirc$ operation (F,V) and (G,Y) is the soft set $(H,\mathbb{Z})$, denoted by $(F,K)\circledcirc_\varepsilon(G,Y)=(H,\mathbb{Z})$, where $\mathbb{Z}=V\cup Y$ and for all $\aleph\in\mathbb{Z}$,

$$H(v)=\begin{cases}F(v), & v\in V-Y\\ G(v), & v\in Y-V\\ F(v)\circledcirc G(v), & v\in V\cap Y\end{cases}$$

(Ali et al., 2009; Aybek, 2024; Maji et al., 2003; Sezgin et al., 2019; Stojanovic, 2021).

Definition 2.7. Let $(F,V),(G,Y)\in S_E(U)$. The complementary extended $\circledcirc$ operation (F,V) and (G,Y) is the soft set $(H,\mathbb{Z})$, denoted by $(F,K)\overset{*}{\circledcirc}_\varepsilon(G,Y)=(H,\mathbb{Z})$, where $\mathbb{Z}=V\cup Y$ and for all $\aleph\in\mathbb{Z}$,

$$H(v)=\begin{cases}F'(v), & v\in V-Y\\ G'(v), & v\in Y-V\\ F(v)\circledcirc G(v), & v\in V\cap Y\end{cases}$$

(Akbulut, 2024; Demirci, 2024; Sarıalioğlu, 2024).

Definition 2.8. Let $(F,V),(G,Y)\in S_E(U)$. The soft binary piecewise $\circledcirc$ of (F,V) and (G,Y) is the soft set (H,V), denoted by $(F,V)\underset{\circledcirc}{\sim}(G,Y)=(H,V)$, where for all $\aleph\in V$,

$$H(v)=\begin{cases}F(v), & v\in V-Y\\ F(v)\circledcirc G(v), & v\in V\cap Y\end{cases}$$

(Eren & Çalışıcı, 2019; Sezgin & Çalışıcı, 2024; Sezgin & Yavuz, 2023b; Yavuz, 2024).

Definition 2.9. Let $(F,V),(G,Y) \in S_E(U)$. The complementary soft binary piecewise ⊙ of (F,V) and (G,Y) is the soft set (H,V), denoted by $(F,V) \underset{\odot}{\overset{*}{\sim}} (G,Y) = (H,V)$, where for all $\aleph \in V$,

$$H(v) = \begin{cases} F'(v), & v \in V - Y \\ F(v) \odot G(v), & v \in V \cap Y \end{cases}$$

(Sezgin & Atagün, 2023; Sezgin & Aybek, 2023; Sezgin et al., 2023a; Sezgin et al., 2023b; Sezgin & Çağman, 2024; Sezgin & Dagtoros, 2023; Sezgin & Demirci, 2023; Sezgin & Sarıalioğlu, 2024; Sezgin & Yavuz, 2023a).

Definition 2.10. Let (S, ⊙) be an algebraic structure. An element $s \in S$ is called idempotent if $s^2=s$. If $s^2=s$ for all $s\in S$, then the algebraic structure $(S,\star)$ is said to be idempotent. An idempotent semigroup is called a band, an idempotent and commutative semigroup is called a semilattice, and an idempotent and commutative monoid is called a bounded semilattice (Clifford, 1954).

Definition 2.11. Let S be a non-empty set, and let "+" and "⊙" be two binary operations defined on S. If the algebraic structure $(S, +, \star)$ satisfies the following properties, then it is called a semiring:

i. (S, +) is a semigroup,
ii. (S, ⊙) is a semigroup,
iii. $x\odot(y + z) = x\odot y + x\odot z$ and $(x +y) \odot z = x\odot z + y\odot z$ for all $x, y, z \in S$.

If $x+y=y+z$ for all $x,y\in S$, then S is called an additive commutative semiring. If $x\odot y=yx$ for all $x,y\in S$, then S is called a multiplicative commutative semiring. If there exists an element $1\in S$ such that $x\odot 1=1\odot x=x$ for all $x\in S$ (multiplicative identity), then S is called semiring with unity. If there exists $0\in S$ such that $0\odot x=x\odot 0=0$ and $0+x=x+0=x$ for all $x\in S$, then 0 is called the zero of S. A semiring with commutative addition and a zero element is called a hemiring (Vandiver, 1934).

Definition 2.12. Let L be a non-empty set, and let "∨" and "∧" be two binary operations defined on L. If the algebraic structure. $(L,\vee,\wedge)$ satisfies the following properties, then it is called a lattice:

i. $(L, \vee)$ a commutative, idempotent semigroup (semilattice)
ii. $(L, \wedge)$ a commutative, idempotent semigroup (semilattice)
iii. $\gamma\vee(\gamma\wedge y)=\gamma\wedge(\gamma\vee y)$, for all $\gamma,y\in L$ (absorption law)
iv.

A lattice with an identity element according to both operations is called a bounded lattice. In a bounded lattice, the identity element of L with respect to the ∧ operation is usually denoted by 1, while the identity element with respect to the ∨ operation is denoted by 0. If the bounded lattice L has an element γ' such that $\gamma\wedge\gamma' = 0$ and $\gamma\vee\gamma'=1$ for all $\gamma\in L$, then L is called a complemented lattice. A lattice holding distribution law is called a distributive lattice. A lattice that is bounded, distributive, and at the same time complemented is called a Boolean algebra. The lattice with De Morgan's laws, i.e., $(\gamma \vee y)' = \gamma'\wedge y'$ and $(\gamma \wedge y)' = \gamma' \vee y'$ for all $\gamma,y\in L$ is called De Morgan algebra (Birkhoff, 1967).

Definition 2.13. Let M be a non-empty set with the binary operation "⊙" and the unary operation "*" defined on M. If 0 is a constant that satisfies the following axioms for any $\gamma, y\in M$, then the structure $(M, \odot, *, 0)$ is called an MV-algebra:

i. $(M, \oplus, 0)$ is a commutative monoid
ii. $(\gamma^*)^*=\gamma$
iii. $0^* \odot\gamma=0^*$
iv. $(\gamma^* \odot y)^* \odot y=(y^* \odot \gamma)^* \odot \gamma$

(Chang, 1959).

3. Complementary Extended Union Operation

In this section, a new soft set operation called the complementary extended union operation is introduced, full algebraic properties of the operation are analyzed by comparing its properties with the union operation in classical set theory.

Definition 3.1. Let $(F,Ƶ)$ and (G,B) be two soft sets over U. The complementary extended union operation of $(F,Ƶ)$ and (G,B) is the soft set $(H,Ş)$, denoted by $(F,Ƶ) \overset{*}{\cup}_{\varepsilon} (G,B) = (H,Ş)$, where for all $\aleph \in Ş = Ƶ \cup B$,

$$H(\aleph)=\begin{cases} F'(\aleph), & \aleph \in Ƶ\text{-}B \\ G'(\aleph), & \aleph \in B\text{-}Ƶ \\ F(\aleph)\cup G(\aleph), & \aleph \in Ƶ \cap B \end{cases}$$

Example 3.2. Let E=$\{e_1,e_2,e_3,e_4\}$ be the parameter set and Ƶ=$\{e_1, e_3\}$ and B=$\{e_2, e_3, e_4\}$ be two subsets of E, and U=$\{h_1,h_2,h_3,h_4,h_5\}$ be the universal set.
Assume that (F,Ƶ)=$\{(e_1,\{h_2,h_5\}),(e_3,\{h_1,h_2,h_5\})\}$,(G,B)=$\{(e_2,\{h_1,h_4,h_5\}),(e_3,\{h_2,h_3,h_4\}),(e_4,\{h_3,h_5\})\}$ be soft sets over U. Let $(F,Ƶ) \overset{*}{\cup}_{\varepsilon}$(G,B)=(H,Ƶ∪B), where for all $\aleph \in Ƶ\cup B$,

$$H(\aleph)=\begin{cases} F'(\aleph), & \aleph \in Ƶ\text{-}B \\ G'(\aleph), & \aleph \in B\text{-}Ƶ \\ F(\aleph)\cup G(\aleph), & \aleph \in Ƶ \cap B \end{cases}$$

Here, since Ƶ∪B=$\{e_1,e_2,e_3,e_4\}$, Ƶ-B=$\{e_1\}$, B-Ƶ=$\{e_2,e_4\}$, Ƶ∩B=$\{e_3\}$,
$H(e_1)=F'(e_1)=\{h_1,h_3,h_4\}$, $H(e_2)=G'(e_2)=\{h_2,h_3\}$, $H(e_4)=G'(e_4)=\{h_1,h_2,h_4\}$ and $H(e_3)=F(e_3)\cup G(e_3)=\{h_1,h_2,h_5\}\cup\{h_2,h_3,h_4\}=\{h_1,h_2,h_3,h_4,h_5\}$.
Thus, $(F,Ƶ) \overset{*}{\cup}_{\varepsilon}$(G,B)=$\{(e_1,\{h_1,h_3,h_4\}), (e_2,\{h_2,h_3\}),(e_3,\{h_1,h_2,h_3,h_4,h_5\}),(e_4,\{h_1,h_2,h_4\})\}$

Theorem 3.3. Algebraic Properties of Operation

1) $\overset{*}{\cup}_{\varepsilon}$ is closed in $S_E(U)$.

Proof: It is clear that $\overset{*}{\cup}_{\varepsilon}$ is a binary operation in $S_E(U)$. Indeed,

$$\overset{*}{\cup}_{\varepsilon}: S_E(U) \times S_E(U) \rightarrow S_E(U)$$

$$((F,Ƶ),(G,B)) \rightarrow (F,Ƶ) \overset{*}{\cup}_{\varepsilon} (G,B)=(H,Ƶ\cup B)$$

Similarly,

$$\overset{*}{\cup}_{\varepsilon}: S_Ƶ(U) \times S_Ƶ(U) \rightarrow S_Ƶ(U)$$

$$((F,Ƶ),(G,Ƶ)) \rightarrow (F,Ƶ) \overset{*}{\cup}_{\varepsilon} (G,Ƶ)=(T,Ƶ\cup Ƶ)=(T,Ƶ)$$

That is, when Ƶ is a fixed subset of the set E and (F, Ƶ) and (G, Ƶ) be elements of $S_Ƶ(U)$, then so is $(F,Ƶ) \overset{*}{\cap}_{\varepsilon}$ (G, Ƶ). Namely, $S_Ƶ$ is closed under $\overset{*}{\cup}_{\varepsilon}$ either.

2) $[(F,Ƶ) \overset{*}{\cup}_{\varepsilon} (G,B)] \overset{*}{\cup}_{\varepsilon} (H,Ş) \neq (F,Ƶ) \overset{*}{\cup}_{\varepsilon} [(G,B) \overset{*}{\cup}_{\varepsilon} (H,Ş)]$.

Proof: First, let's consider the left-hand side (LHS). Suppose $(F,Ƶ) \overset{*}{\cup}_{\varepsilon}$ (G,B)=(T,Ƶ∪B), where for all $\aleph \in Ƶ\cup B$,

$$T(ℵ)=\begin{cases} F'(ℵ), & ℵ\in Z-B \\ G'(ℵ), & ℵ\in B-Z \\ F(ℵ)\cup G(ℵ), & ℵ\in Z\cap B \end{cases}$$

Let (T,Z∪B) $\overset{*}{\cup}_{\varepsilon}$ (H,Ş) = (M,Z∪B∪Ş), where for all ℵ∈Z∪B∪Ş,

$$M(ℵ)=\begin{cases} T'(ℵ), & ℵ\in (Z\cup B)-Ş \\ H'(ℵ), & ℵ\in Ş-(Z\cup B) \\ T(ℵ)\cup H(ℵ), & ℵ\in (Z\cup B)\cap Ş \end{cases}$$

Hence,

$$M(ℵ)=\begin{cases} F(ℵ), & ℵ\in (Z-B)-Ş=Z\cap B'\cap Ş' \\ G(ℵ), & ℵ\in (B-Z)-Ş=Z'\cap B\cap Ş' \\ F'(ℵ)\cap G'(ℵ), & ℵ\in (Z\cap B)-Ş=Z\cap B\cap Ş' \\ H'(ℵ) & ℵ\in Ş-(Z\cup B)=Z'\cap B'\cap Ş \\ F'(ℵ)\cup H(ℵ), & ℵ\in (Z-B)\cap Ş=Z\cap B'\cap Ş \\ G'(ℵ)\cup H(ℵ), & ℵ\in (B-Z)\cap Ş=Z'\cap B\cap Ş \\ \big(F(ℵ)\cup G(ℵ)\big)\cup H(ℵ), & ℵ\in (Z\cap B)\cap Ş=Z\cap B\cap Ş \end{cases}$$

Now consider the RHS, i.e., (F, Z) $\overset{*}{\cup}_{\varepsilon}$[(G,B) $\overset{*}{\cup}_{\varepsilon}$(H,Ş)]. Let (G,B) $\overset{*}{\cup}_{\varepsilon}$ (H,Ş)=(K,B∪Ş). So, for all ℵ∈B∪Ş,

$$K(ℵ)=\begin{cases} G'(ℵ), & ℵ\in B-Ş \\ H'(ℵ), & ℵ\in Ş-B \\ G(ℵ)\cup H(ℵ), & ℵ\in B\cap Ş \end{cases}$$

Let (F, Z) $\overset{*}{\cup}_{\varepsilon}$(K,B∪Ş) = (S,Z∪B∪Ş). Thus, for all ℵ∈Z∪B∪Ş,

$$S(ℵ)=\begin{cases} F'(ℵ), & ℵ\in Z-(B\cup Ş) \\ K'(ℵ), & ℵ\in (B\cup Ş)-Z \\ F(ℵ)\cup K(ℵ), & ℵ\in Z\cap (B\cup Ş) \end{cases}$$

Hence,

$$S(ℵ)=\begin{cases} F'(ℵ), & ℵ\in Z-(B\cup Ş)=Z\cap B'\cap Ş' \\ G(ℵ), & ℵ\in (B-Ş)-Z=Z'\cap B\cap Ş' \\ H(ℵ), & ℵ\in (Ş-B)-Z=Z'\cap B'\cap Ş \\ G'(ℵ)\cap H'(ℵ), & ℵ\in (B\cap Ş)-Z=Z'\cap B\cap Ş \\ F(ℵ)\cup G'(ℵ), & ℵ\in Z\cap (B-Ş)=Z\cap B\cap Ş' \\ F(ℵ)\cup H'(ℵ), & ℵ\in Z\cap (Ş-B)=Z\cap B'\cap Ş \\ F(ℵ)\cup (G(ℵ)\cup H(ℵ)), & ℵ\in Z\cap (B\cap Ş)=Z\cap B\cap Ş \end{cases}$$

Thus, M≠S. That is, in the set $S_E(U)$, $\overset{*}{\cup}_{\varepsilon}$ is not associative. However, we have the following:

3) [(F, Z) $\overset{*}{\cup}_{\varepsilon}$(G,Z)] $\overset{*}{\cup}_{\varepsilon}$(H,Z) = (F, Z) $\overset{*}{\cup}_{\varepsilon}$[(G, Z) $\overset{*}{\cup}_{\varepsilon}$(H,Z)].

4) (F, Z) $\overset{*}{\cup}_{\varepsilon}$(G,B)=(G,B) $\overset{*}{\cup}_{\varepsilon}$ (F,Z).

Proof: Firstly, the parameter sets of the soft set on both sides of the equation is Z∪B, and thus the first condition of the soft equality is satisfied. Now let's consider the LHS. Let (F, Z) $\overset{*}{\cup}_{\varepsilon}$(G,B)=(H,Z∪B), where for all ℵ∈Z∪B,

$$H(ℵ)=\begin{cases} F'(ℵ), & ℵ\in Z-B \\ G'(ℵ), & ℵ\in B-Z \\ F(ℵ)\cup G(ℵ), & ℵ\in Z\cap B \end{cases}$$

Now consider the RHS, i.e., (G,B) $\overset{*}{\cup}_{\varepsilon}$(F,Z). Let (G,B) $\overset{*}{\cup}_{\varepsilon}$ (F,Z)=(T,B∪Z), where for all ℵ∈B∪Z,

$$T(\aleph)=\begin{cases} G'(\aleph), & \aleph\in B-Z \\ F'(\aleph), & \aleph\in Z-B \\ G(\aleph)\cup F(\aleph), & \aleph\in B\cap Z \end{cases}$$

Thus, it is seen that H=T. Similarly, one can show that (F, Z) $\overset{*}{\cup}_{\varepsilon}$(G,Z)=(G, Z) $\overset{*}{\cup}_{\varepsilon}$(F,Z. That is, $\overset{*}{\cap}_{\varepsilon}$ operation is commutative in both $S_E(U)$ and $S_Z(U)$.

5) (F, Z) $\overset{*}{\cup}_{\varepsilon}$(F,Z)=(F,Z).

Proof: Let (F, Z) $\overset{*}{\cup}_{\varepsilon}$ (F,Z)=(H,Z∪Z), where for all ℵ∈Z,

$$H(\aleph)=\begin{cases} F'(\aleph), & \aleph\in Z-Z=\emptyset \\ F'(\aleph), & \aleph\in Z-Z=\emptyset \\ F(\aleph)\cup F(\aleph), & \aleph\in Z\cap Z=Z \end{cases}$$

Hence, H(ℵ)= F(ℵ)∪F(ℵ)=F(ℵ), for all ℵ∈Z. Thus, (H,Z)=(F,Z). That is, $\overset{*}{\cup}_{\varepsilon}$ is idempotent in $S_E(U)$.

6) (F, Z) $\overset{*}{\cup}_{\varepsilon}$ $\emptyset_Z$=$\emptyset_Z$ $\overset{*}{\cup}_{\varepsilon}$(F,Z)=(F,Z).

Proof: Let $\emptyset_Z$ =(S,Z). Then, for all ℵ∈Z, S(ℵ)= ∅. Let (F, Z) $\overset{*}{\cup}_{\varepsilon}$ (S,Z)=(H,Z∪Z), where for all ℵ∈Z,

$$H(\aleph)=\begin{cases} F'(\aleph), & \aleph\in Z-Z=\emptyset \\ S'(\aleph), & \aleph\in Z-Z=\emptyset \\ F(\aleph)\cup S(\aleph), & \aleph\in Z\cap Z=Z \end{cases}$$

Hence, H(ℵ) = F(ℵ)∪S(ℵ)= F(ℵ)∪∅=F(ℵ), for all ℵ∈Z. Thus, (H,Z)=(F,Z). That is, in $S_Z(U)$, the identity element of $\overset{*}{\cup}_{\varepsilon}$ is the soft set $\emptyset_Z$.

Corollary 3.3.1. By Theorem 3.3 (1), (3), (4), (5), and (6), ($S_Z(U)$, $\overset{*}{\cup}_{\varepsilon}$) is a commutative, idempotent monoid, that is, a bounded semilattice, whose identity element is $\emptyset_Z$, where Z ⊆ E is a fixed set of parameters. Moreover, from Theorem 3.3 (2), $\overset{*}{\cup}_{\varepsilon}$ cannot form a semigroup as it is not associative in S_E (U). Thus, ($S_E(U)$, $\overset{*}{\cup}_{\varepsilon}$) is a groupoid.

7) (F, Z) $\overset{*}{\cup}_{\varepsilon}$ U_Z=U_Z $\overset{*}{\cup}_{\varepsilon}$(F,Z)= U_Z.

Proof: Let U_Z =(T,Z). Thus, T(ℵ)=U for all ℵ∈Z. Let (F, Z) $\overset{*}{\cup}_{\varepsilon}$(T, A) =(H,Z∪Z), where for all ℵ∈Z,

$$H(\aleph)=\begin{cases} F'(\aleph), & \aleph\in Z-Z=\emptyset \\ T'(\aleph), & \aleph\in Z-Z=\emptyset \\ F(\aleph)\cup T(\aleph), & \aleph\in Z\cap Z=Z \end{cases}$$

Hence, H(ℵ) = F(ℵ)∪T(ℵ)= F(ℵ)∪U=U, for all ℵ∈Z, and (H,Z)= U_Z. That is, the absorbing element of $\overset{*}{\cup}_{\varepsilon}$ in $S_Z(U)$ is the soft set U_Z .

8) (F, Z) $\overset{*}{\cap}_{\varepsilon}$ $\emptyset_\emptyset$= $\emptyset_\emptyset$ $\overset{*}{\cap}_{\varepsilon}$(F,Z) = $(F,Z)^r$.

Proof: Let $\emptyset_\emptyset$= (K, ∅) and (F, Z) $\overset{*}{\cup}_{\varepsilon}$(K, ∅) = (Q,Z∪∅) = (Q, Z), where for all ℵ∈ Z,

$$Q(\aleph)=\begin{cases} F'(\aleph), & \aleph\in Z\text{-}\emptyset=Z \\ K'(\aleph), & \aleph\in\emptyset\text{-}Z=\emptyset \\ F(\aleph)\cup K(\aleph), & \aleph\in Z\cap\emptyset=\emptyset \end{cases}$$

Hence, $Q(\aleph) = F'(\aleph)$, for all $\aleph\in Z$, so $(Q,Z) = (F,Z)^r$.

9) $(F, Z) \overset{*}{\cup_\varepsilon} (F,Z)^r = (F,Z)^r \overset{*}{\cup_\varepsilon} (F,Z)=U_Z$.

Proof: Let $(F,Z)^r = (H,Z)$, hence $H(\aleph)=F'(\aleph)$ for all $\aleph\in Z$. Let $(F, Z) \overset{*}{\cup_\varepsilon} (H,Z)=(T,Z\cup Z)$, where for all $\aleph\in Z$,

$$T(\aleph)=\begin{cases} F'(\aleph), & \aleph\in Z\text{-}Z=\emptyset \\ H'(\aleph), & \aleph\in Z\text{-}Z=\emptyset \\ F(\aleph)\cup H(\aleph), & \aleph\in Z\cap Z=Z \end{cases}$$

Here, $T(\aleph)= F(\aleph)\cup H(\aleph)= F(\aleph) \cup F'(\aleph)=U$ for all $\aleph\in Z$. Thus, $(T,Z)= U_Z$.

10) $[(F, Z) \overset{*}{\cup_\varepsilon} (G,B)]^r =(F,Z)^r \overset{*}{\cap_\varepsilon} (G,B)^r$

Proof: Let $(F, Z) \overset{*}{\cup} (G,B)=(H,Z\cup B)$, where for all $\aleph\in Z\cup B$,

$$H(\aleph)=\begin{cases} F'(\aleph), & \aleph\in Z\text{-}B \\ G'(\aleph), & \aleph\in B\text{-}Z \\ F(\aleph)\cup G(\aleph), & \aleph\in Z\cap B \end{cases}$$

Let $(H,Z\cup B)^r =(T,Z\cup B)$, where for all $\aleph\in Z\cup B$,

$$T(\aleph)=\begin{cases} F(\aleph), & \aleph\in Z\text{-}B \\ G(\aleph), & \aleph\in B\text{-}Z \\ F'(\aleph)\cap G'(\aleph), & \aleph\in Z\cap B \end{cases}$$

Hence, $(T,Z\cup B) =(F,Z)^r \overset{*}{\cap_\varepsilon} (G,B)^r$

11) $(F, Z) \overset{*}{\cup_\varepsilon} (G, Z)= \emptyset_Z \Leftrightarrow (F, Z) = (G, Z) = \emptyset_Z$.

Proof: Let $(F, Z) \overset{*}{\cup_\varepsilon} (G, Z) = T,(Z\cup Z)$, where for all $\aleph\in Z$,

$$T(\aleph)=\begin{cases} F'(\aleph), & \aleph\in Z\text{-}Z=\emptyset \\ G'(\aleph), & \aleph\in Z\text{-}Z=\emptyset \\ F(\aleph)\cup G(\aleph), & \aleph\in Z\cap Z=Z \end{cases}$$

Since $(T,Z)= \emptyset_Z$, $T(\aleph)=F(\aleph)\cup G(\aleph)=\emptyset$, for all $\aleph \in Z$. Hence, $F(\aleph)=G(\aleph)=\emptyset$ for all $\aleph \in Z$. Thus, $(F, Z) = (G, Z) = \emptyset_Z$.

12) $\emptyset_Z \tilde{\subseteq}(F, Z) \overset{*}{\cup_\varepsilon} (G,B)$, $\emptyset_B \tilde{\subseteq}(F, Z) \overset{*}{\cup_\varepsilon} (G,B)$, $\emptyset_{Z\cup B} \tilde{\subseteq}(F, Z) \overset{*}{\cup_\varepsilon} (G,B)$, $(F, Z) \overset{*}{\cup_\varepsilon} (G,B) \tilde{\subseteq} U_{Z\cup B}$.

13) $(F,Z) \tilde{\subseteq} (F, Z) \overset{*}{\cup_\varepsilon} (G,Z)$ and $(G,Z) \tilde{\subseteq} (F, Z) \overset{*}{\cup_\varepsilon} (G,Z)$.

Proof: Le $(F, Z) \overset{*}{\cup_\varepsilon} (G,Z)=(H, Z \cup Z)$, where for all $\aleph\in Z$,

$$H(\aleph)=\begin{cases} F'(\aleph), & \aleph\in Z\text{-}Z=\emptyset \\ G'(\aleph), & \aleph\in Z\text{-}Z=\emptyset \\ F(\aleph) \cup G(\aleph), & \aleph\in Z\cap Z=Z \end{cases}$$

Since $H(\aleph)=F(\aleph) \subseteq F(\aleph) \cup G(\aleph)$, for all $\aleph\in Z$, $(F,Z)\tilde{\subseteq}(F, Z) \overset{*}{\cup_\varepsilon} (G,Z)$. Similarly, since $H(\aleph)=G(\aleph) \subseteq F(\aleph) \cup G(\aleph)$, for all $\aleph\in Z$, $(G,Z) \tilde{\subseteq} (F, Z) \overset{*}{\cup_\varepsilon} (G,Z)$.

14) $(F,Z) \tilde{\subseteq} (G, Z) \Leftrightarrow (F, Z) \overset{*}{\cup_\varepsilon} (G,Z) =(G,Z)$.

Proof: Let $(F, Ƶ) \,\widetilde{\subseteq}\, (G, Ƶ)$. Then, $F(\aleph) \subseteq G(\aleph)$ for all $\aleph \in Ƶ$, and so $G'(\aleph) \subseteq F'(\aleph)$. Let $(F, Ƶ) \overset{*}{\cup}_{\varepsilon} (G,Ƶ)=(H,Ƶ\cupƵ)$, where for all $\aleph \in Ƶ$,

$$H(\aleph)=\begin{cases} F'(\aleph), & \aleph \in Ƶ\text{-}Ƶ=\emptyset \\ G'(\aleph), & \aleph \in Ƶ\text{-}Ƶ=\emptyset \\ F(\aleph) \cup G(\aleph), & \aleph \in Ƶ\capƵ=Ƶ \end{cases}$$

Thus, $H(\aleph)=F(\aleph) \cup G(\aleph)=G(\aleph,)$ for all $\aleph \in Ƶ$. Hence, $(F, Ƶ) \overset{*}{\cup}_{\varepsilon}(G,Ƶ)=(G,Ƶ)$.

Conversely, let $(F,Ƶ) \overset{*}{\cup}_{\varepsilon}(G,Ƶ)=(G,Ƶ)$. Hence, $F(\aleph)\cup G(\aleph)=G(\aleph)$, for all $\aleph \in Ƶ$ and thus, $F(\aleph)\subseteq G(\aleph)$. Therefore, $(F,Ƶ) \,\widetilde{\subseteq}\, (G, Ƶ)$.

15) $(F, Ƶ) \overset{*}{\cap}_{\varepsilon}(G, B) \,\widetilde{\subseteq}\, (F, Ƶ) \overset{*}{\cup}_{\varepsilon}(G, B)$.

Proof: Let $(F, Ƶ) \overset{*}{\cap}_{\varepsilon}(G,B)=(H,Ƶ\cup B)$, where $\aleph \in Ƶ\cup B$,

$$H(\aleph)=\begin{cases} F'(\aleph), & \aleph \in Ƶ\text{-}B \\ G'(\aleph), & \aleph \in B\text{-}Ƶ \\ F(\aleph)\cap G(\aleph), & \aleph \in Ƶ\cap B \end{cases}$$

Let $(F, Ƶ) \overset{*}{\cup}_{\varepsilon} (G,B)=(T,Ƶ\cup B)$, where for all $\aleph \in Ƶ\cup B$,

$$T(\aleph)=\begin{cases} F'(\aleph), & \aleph \in Ƶ\text{-}B \\ G'(\aleph), & \aleph \in B\text{-}Ƶ \\ F(\aleph)\cup G(\aleph), & \aleph \in Ƶ\cap B \end{cases}$$

Since $H(\aleph)=F'(\aleph) \subseteq F'(\aleph)=T(\aleph)$, for all $\aleph \in Ƶ\text{-}B$, $H(\aleph)=G'(\aleph)\subseteq G'(\aleph)=T(\aleph)$, for all $\aleph \in B\text{-}Ƶ$, and $F(\aleph) \cap G(\aleph) \subseteq F(\aleph) \cup G(\aleph)$, for all $\aleph \in Ƶ\cap B$, $(F, Ƶ) \overset{*}{\cap}_{\varepsilon}(G, B) \,\widetilde{\subseteq}\, (F, Ƶ) \overset{*}{\cup}_{\varepsilon}(G, B)$.

16) $(F, Ƶ) \overset{*}{\cap}_{\varepsilon}(G, B) = (F, Ƶ) \overset{*}{\cup}_{\varepsilon}(G, B) \Leftrightarrow (F,Ƶ\cap B) = (G,Ƶ\cap B)$.

Proof: Let $(F, Ƶ) \overset{*}{\cap}_{\varepsilon}(G, B) \,\widetilde{\subseteq}\, (F, Ƶ) \overset{*}{\cup}_{\varepsilon}(G, B)$ and $(F, Ƶ) \overset{*}{\cap}_{\varepsilon} (G,B) = (H,Ƶ\cup B)$, where for all $\aleph \in Ƶ\cup B$,

$$H(\aleph)=\begin{cases} F'(\aleph), & \aleph \in Ƶ\text{-}B \\ G'(\aleph), & \aleph \in B\text{-}Ƶ \\ F(\aleph)\cap G(\aleph), & \aleph \in Ƶ\cap B \end{cases}$$

Let $(F, Ƶ) \overset{*}{\cup}_{\varepsilon} (G,B) = (K,Ƶ\cup B)$, where for all $\aleph \in Ƶ\cup B$,

$$K(\aleph)=\begin{cases} F'(\aleph), & \aleph \in Ƶ\text{-}B \\ G'(\aleph), & \aleph \in B\text{-}Ƶ \\ F(\aleph)\cup G(\aleph), & \aleph \in B\capƵ \end{cases}$$

Since $(H,Ƶ\cup B)=(K,Ƶ\cup B)$, $F'(\aleph)=F'(\aleph)$, for all $\aleph \in Ƶ\text{-}B$, $G'(\aleph)=G'(\aleph)$, for all $\aleph \in B\text{-}Ƶ$, and $F(\aleph)\cap G(\aleph)= F(\aleph)\cup G(\aleph)$, for all $\aleph \in Ƶ\cap B$. So, $F(\aleph) = G(\aleph)$ for all $\aleph \in Ƶ \cap B$. Thus, $(F,Ƶ \cap B) = (G, Ƶ \cap B)$.

Conversely, let $(F,Ƶ\cap B)=(G,Ƶ\cap B)$. Hence, $F(\aleph) = G(\aleph)$, for all $\aleph \in Ƶ \cap B$. So, $F(\aleph)\cap G(\aleph)= F(\aleph)\cup G(\aleph)$, for all $\aleph \in Ƶ \cap B$. Moreover, since $F(\aleph)=F'(\aleph)$, for all $\aleph \in Ƶ\text{-}B$, and $G'(\aleph)=G'(\aleph)$, and for all $\aleph \in B\text{-}Ƶ$, $H(\aleph)=K(\aleph)$, for all $\aleph \in Ƶ\cup B$. Thus, $(H,Ƶ\cup B) = (K,Ƶ\cup B)$ and $(F, Ƶ) \overset{*}{\cap}_{\varepsilon}(G, B) = (F, Ƶ) \overset{*}{\cup}_{\varepsilon}(G, B)$.

17) If $(F,Ƶ) \,\widetilde{\subseteq}\,(G,Ƶ)$, then $(F, Ƶ) \overset{*}{\cup}_{\varepsilon}(H, Ƶ) \,\widetilde{\subseteq}\, (G, Ƶ) \overset{*}{\cup}_{\varepsilon}(H, Ƶ)$.

Proof: Let $(F, Ƶ) \,\widetilde{\subseteq}\, (G, Ƶ)$. Hence, $F(\aleph)\subseteq G(\aleph)$, for all $\aleph \in Ƶ$. Let $(F, Ƶ) \overset{*}{\cup}_{\varepsilon}(H,Ƶ)=(W,Ƶ)$. Thus, for all $\aleph \in Ƶ$,

$$W(\aleph)=\begin{cases} F'(\aleph), & \aleph \in Ƶ\text{-}Ƶ=\emptyset \\ H'(\aleph), & \aleph \in Ƶ\text{-}Ƶ=\emptyset \\ F(\aleph)\cup H(\aleph), & \aleph \in Ƶ\capƵ=Ƶ \end{cases}$$

Let $(\mathrm{G}, \mathrm{Z}) \overset{*}{\cup_{\varepsilon}} (\mathrm{H}, \mathrm{Z}) = (\mathrm{L}, \mathrm{Z})$. Thus, for all $\aleph \in \mathrm{Z}$,

$$\mathrm{L}(\aleph)=\begin{cases} \mathrm{G}'(\aleph), & \aleph\in \mathrm{Z}\text{-}\mathrm{Z}=\emptyset \\ \mathrm{H}'(\aleph), & \aleph\in \mathrm{Z}\text{-}\mathrm{Z}=\emptyset \\ \mathrm{G}(\aleph)\cup \mathrm{H}(\aleph), & \aleph\in \mathrm{Z}\cap \mathrm{Z}=\mathrm{Z} \end{cases}$$

Thus, $\mathrm{W}(\aleph)=\mathrm{F}(\aleph)\cup\mathrm{H}(\aleph) \subseteq \mathrm{G}(\aleph)\cup\mathrm{H}(\aleph)=\mathrm{L}(\aleph)$, for all $\aleph\in\mathrm{Z}$. Hence, $(\mathrm{F},\mathrm{Z}) \overset{*}{\cup_{\varepsilon}} (\mathrm{H},\mathrm{Z}) \tilde{\subseteq} (\mathrm{G},\mathrm{Z}) \overset{*}{\cup_{\varepsilon}} (\mathrm{H},\mathrm{Z})$.

18) If $(\mathrm{F}, \mathrm{Z}) \overset{*}{\cup_{\varepsilon}} (\mathrm{H}, \mathrm{Z}) \tilde{\subseteq} (\mathrm{G}, \mathrm{Z}) \overset{*}{\cup_{\varepsilon}} (\mathrm{H}, \mathrm{Z})$, then $(\mathrm{F},\mathrm{Z}) \tilde{\subseteq} (\mathrm{G},\mathrm{Z})$ needs not be true. That is, the converse of Theorem 3.3 (17) is not true.

Proof: Let us give an example to show that the converse of Theorem 3.3 (17) is not true. Let $E=\{e_1,e_2,e_3,e_4,e_5\}$ be the parameter set, $\mathrm{Z}=\{e_1,e_3\}$ be a subset of E, and $U=\{h_1,h_2,h_3,h_4,h_5\}$ be universal set, (F,Z), (G,Z) and (H,Z) be soft sets over U such that $(\mathrm{F},\mathrm{Z})=\{(e_1,\{h_2,h_5\}), (e_3,\{h_1,h_2,h_5\})\}$, $(\mathrm{G},\mathrm{Z})=\{(e_1,\{h_2\}),(e_3,\{h_1,h_2\})\}$, $(\mathrm{H},\mathrm{Z})=\{(e_1,U),(e_3,U)\}$

Let $(\mathrm{F},\mathrm{Z}) \overset{*}{\cup_{\varepsilon}} (\mathrm{H},\mathrm{Z})=(\mathrm{L},\mathrm{Z})$, thus $(\mathrm{L},\mathrm{Z})=\{(e_1,U),(e_3,U)\}$ and let $(\mathrm{G}, \mathrm{Z}) \overset{*}{\cup_{\varepsilon}} (\mathrm{H}, \mathrm{Z})=(\mathrm{K},\mathrm{Z})$, thus $(\mathrm{K},\mathrm{Z})=\{(e_1,U),(e_3,U)\}$. Hence, $(\mathrm{F}, \mathrm{Z}) \overset{*}{\cup_{\varepsilon}} (\mathrm{H}, \mathrm{Z}) \tilde{\subseteq} (\mathrm{G}, \mathrm{Z}) \overset{*}{\cup_{\varepsilon}} (\mathrm{H}, \mathrm{Z})$ but it is obvious that $(\mathrm{F},\mathrm{Z}) \tilde{\subseteq} (\mathrm{G},\mathrm{Z})$ is not satisfied.

19) If $(\mathrm{F},\mathrm{Z}) \tilde{\subseteq} (\mathrm{G}, \mathrm{B})$ and $(\mathrm{K},\mathrm{Z}) \tilde{\subseteq} (\mathrm{L}, \mathrm{B})$, then $(\mathrm{F}, \mathrm{Z}) \overset{*}{\cup_{\varepsilon}} (\mathrm{K},\mathrm{Z}) \tilde{\subseteq} (\mathrm{G},\mathrm{B}) \overset{*}{\cup_{\varepsilon}} (\mathrm{L},\mathrm{B})$.

Proof: Let $(\mathrm{F}, \mathrm{Z}) \tilde{\subseteq} (\mathrm{G}, \mathrm{B})$ and $(\mathrm{K},\mathrm{Z}) \tilde{\subseteq} (\mathrm{L}, \mathrm{B})$. Thus, $\mathrm{Z} \subseteq \mathrm{B}$ and for all $\aleph\in\mathrm{Z}$, $\mathrm{F}(\aleph)\subseteq \mathrm{G}(\aleph)$ and $\mathrm{K}(\aleph)\subseteq \mathrm{L}(\aleph)$. Let $(\mathrm{F}, \mathrm{Z}) \overset{*}{\cup_{\varepsilon}} (\mathrm{K},\mathrm{Z}) = (\mathrm{W},\mathrm{Z})$. Thus, for all $\aleph\in\mathrm{Z}$,

$$\mathrm{W}(\aleph)=\begin{cases} \mathrm{F}'(\aleph), & \aleph\in \mathrm{Z}\text{-}\mathrm{Z}=\emptyset \\ \mathrm{K}'(\aleph), & \aleph\in \mathrm{Z}\text{-}\mathrm{Z}=\emptyset \\ \mathrm{F}(\aleph) \cup\mathrm{K}(\aleph), & \aleph\in \mathrm{Z}\cap \mathrm{Z}=\mathrm{Z} \end{cases}$$

Let $(\mathrm{G},\mathrm{B}) \overset{*}{\cup_{\varepsilon}} (\mathrm{L}, \mathrm{B}) = (\mathrm{S},\mathrm{B})$. Thus, for all $\aleph\in\mathrm{B}$,

$$\mathrm{S}(\aleph)=\begin{cases} \mathrm{G}'(\aleph), & \aleph\in \mathrm{B}\text{-}\mathrm{B}=\emptyset \\ \mathrm{L}'(\aleph), & \aleph\in \mathrm{B}\text{-}\mathrm{B}=\emptyset \\ \mathrm{G}(\aleph) \cup\mathrm{L}(\aleph), & \aleph\in \mathrm{B}\cap \mathrm{B}=\mathrm{B} \end{cases}$$

Hence, for all $\aleph \in\mathrm{Z}$, $\mathrm{W}(\aleph)=\mathrm{F}(\aleph) \cup\mathrm{K}(\aleph) \subseteq \mathrm{G}(\aleph) \cup\mathrm{L}(\aleph)=\mathrm{S}(\aleph)$ and so $(\mathrm{F}, \mathrm{Z}) \overset{*}{\cup_{\varepsilon}} (\mathrm{K},\mathrm{Z}) \tilde{\subseteq} (\mathrm{G},\mathrm{B}) \overset{*}{\cup_{\varepsilon}} (\mathrm{L},\mathrm{B})$.

20) $(\mathrm{F}, \mathrm{Z}) \overset{*}{\cap_{\varepsilon}} ([(\mathrm{F},\mathrm{Z}) \overset{*}{\cup_{\varepsilon}} (\mathrm{G},\mathrm{Z})]= (\mathrm{F}, \mathrm{Z})$ and $(\mathrm{F},\mathrm{Z}) \overset{*}{\cup_{\varepsilon}} [(\mathrm{F},\mathrm{Z}) \overset{*}{\cap_{\varepsilon}} (\mathrm{G},\mathrm{Z})] = (\mathrm{F},\mathrm{Z})$ (absorbtion laws)

Proof: Let's consider the LHS. Let $(\mathrm{F}, \mathrm{Z}) \overset{*}{\cup_{\varepsilon}} (\mathrm{G},\mathrm{Z})=(\mathrm{T},\mathrm{Z})$, where for all $\aleph\in\mathrm{Z}$,

$$\mathrm{T}(\aleph)=\begin{cases} \mathrm{F}'(\aleph), & \aleph\in \mathrm{Z}\text{-}\mathrm{Z}=\emptyset \\ \mathrm{G}'(\aleph), & \aleph\in \mathrm{Z}\text{-}\mathrm{Z}=\emptyset \\ \mathrm{F}(\aleph)\cap\mathrm{G}(\aleph), & \aleph\in \mathrm{Z}\cap \mathrm{Z}=\mathrm{Z} \end{cases}$$

Let $(\mathrm{F}, \mathrm{Z}) \overset{*}{\cap_{\varepsilon}} (\mathrm{T},\mathrm{Z})=(\mathrm{M},\mathrm{Z})$, where for all $\aleph\in\mathrm{Z}$,

$$\mathrm{M}(\aleph)=\begin{cases} \mathrm{F}'(\aleph), & \aleph\in \mathrm{Z}\text{-}\mathrm{Z}=\emptyset \\ \mathrm{T}'(\aleph), & \aleph\in \mathrm{Z}\text{-}\mathrm{Z}=\emptyset \\ \mathrm{F}(\aleph)\cap\mathrm{T}(\aleph), & \aleph\in \mathrm{Z}\cap \mathrm{Z}=\mathrm{Z} \end{cases}$$

Thus,

$$\mathrm{M}(\aleph)=\begin{cases} \mathrm{F}'(\aleph), & \aleph\in \mathrm{Z}\text{-}\mathrm{Z}=\emptyset \\ \mathrm{F}(\aleph) \cup [\mathrm{F}(\aleph) \cap \mathrm{G}(\aleph)] & \aleph\in \mathrm{Z}\cap \mathrm{Z}=\mathrm{Z} \end{cases}$$

Hence,

$$M(\aleph)=\begin{cases} F'(\aleph), & \aleph\in Ƶ-Ƶ=\emptyset \\ F(\aleph), & \aleph\in Ƶ\cap Ƶ=Ƶ \end{cases}$$

Thus, $(M,Ƶ)=(F,Ƶ)$ and so $(F,Ƶ) \overset{*}{\cup}_{\varepsilon}[(F,Ƶ) \overset{*}{\cap}_{\varepsilon}(G,Ƶ)] = (F,Ƶ)$.

Similarly, $(F,Ƶ) \overset{*}{\cup}_{\varepsilon}[(F,Ƶ) \overset{*}{\cap}_{\varepsilon}(G,Ƶ)] = (F,Ƶ)$ can be shown. Here, if $\overset{*}{\cap}_{\varepsilon}$ is replaced by the restricted intersection operation or the extended intersection operation, it is evident that Theorem 3.3. (20) holds again in $S_Ƶ$ (U), since these operations coincide with each other in the collection $S_Ƶ$(U).

Theorem 3.4. $(S_Ƶ(U),^r, \overset{*}{\cup}_{\varepsilon},\emptyset_Ƶ)$ is an MV-algebra.

Proof: Let us show that it satisfies the MV-algebra conditions.

- (MV1) $(S_Ƶ(U), \overset{*}{\cup}_{\varepsilon}, \emptyset_Ƶ)$ is a commutative monoid (Corollary 3.3.1).
- (MV2) $((F,Ƶ)^r)^r=(F,Ƶ)$ (Ali et al., 2011).
- (MV3) $(\emptyset_Ƶ)^r \overset{*}{\cup}_{\varepsilon}(F,Ƶ) = U_Ƶ \overset{*}{\cup}_{\varepsilon}(F,Ƶ) = U_Ƶ = (\emptyset_Ƶ)^r$.
- (MV4) $[(F,Ƶ)^r \overset{*}{\cup}_{\varepsilon}(G,Ƶ)]^r \overset{*}{\cup}_{\varepsilon}(G,Ƶ)=((G,Ƶ)^r \overset{*}{\cup}_{\varepsilon}(F,Ƶ)^r)^r \overset{*}{\cup}_{\varepsilon}(F,Ƶ)$. Indeed,
  $[(F,Ƶ)^r \overset{*}{\cup}_{\varepsilon}(G,Ƶ)]^r \overset{*}{\cup}_{\varepsilon}(G,Ƶ)=[((F,Ƶ)^r)^r \overset{*}{\cap}_{\varepsilon}(G,Ƶ)^r] \overset{*}{\cup}_{\varepsilon}(G,Ƶ)=[(F,Ƶ) \overset{*}{\cap}_{\varepsilon}(G,Ƶ)^r] \overset{*}{\cup}_{\varepsilon}(G,Ƶ)=[(F,Ƶ) \overset{*}{\cup}_{\varepsilon}(G,Ƶ)] \overset{*}{\cap}_{\varepsilon}[(G,Ƶ)^r \overset{*}{\cup}_{\varepsilon}(G,Ƶ)]=[(F,Ƶ) \overset{*}{\cup}_{\varepsilon}(G,Ƶ)] \overset{*}{\cap}_{\varepsilon}[(F,Ƶ) \overset{*}{\cup}_{\varepsilon}(F,Ƶ)^r]=(F,Ƶ) \overset{*}{\cup}_{\varepsilon}[(G,Ƶ) \overset{*}{\cap}_{\varepsilon}(F,Ƶ)^r]=(F,Ƶ) \overset{*}{\cup}_{\varepsilon}[(G,Ƶ)^r \overset{*}{\cup}_{\varepsilon}(F,Ƶ)]^r=[(G,Ƶ)^r \overset{*}{\cup}_{\varepsilon}(F,Ƶ)]^r \overset{*}{\cup}_{\varepsilon}(F,Ƶ)$

Thus, $(S_Ƶ(U),^r, \overset{*}{\cup}_{\varepsilon}, \emptyset_Ƶ)$ is an MV-algebra.

4. Distribution Rules

In this section, the distribution rules of complementary extended union operation over other types of soft set operations are studied, and many algebraic structures are obtained in the collection of soft sets with a fixed parameter set with complementary extended soft set operations and other types of soft set operations.

Theorem 4.1. Let $(F,Ƶ)$, (G,B), $(H,Ş)$ be soft sets over U. The complementary extended union operation has the following distributions over restricted soft set operations:

i) LHS Distributions of the Complementary Extended Union Operation over Restricted Soft Set Operations:

1)If $(Ƶ\Delta B)\cap Ş=Ƶ\cap B\cap Ş'=\emptyset$, then $(F,Ƶ)\overset{*}{\cup}_{\varepsilon}[(G,B)\cap_R(H,Ş)]=[(F,Ƶ)\overset{*}{\cup}_{\varepsilon}(G,B)]\cup_R[(F,Ƶ)\overset{*}{\cup}_{\varepsilon}(H,Ş)]$.

Proof: Consider first the LHS. Let $(G,B)\cap_R(H,Ş)=(M,B\cap Ş)$, where for all $\aleph\in B\cap Ş$, $M(\aleph) = G(\aleph) \cap H(\aleph)$. Let $(F,Ƶ) \overset{*}{\cup}_{\varepsilon}(M, B\cap Ş)=(N, Ƶ\cup(B\cap Ş))$, where for all $\aleph\in Ƶ\cup(B\cap Ş)$,

$$N(\aleph)=\begin{cases} F'(\aleph), & \aleph\in Ƶ-(B\cap Ş) \\ M'(\aleph), & \aleph\in(B\cap Ş)-Ƶ \\ F(\aleph)\cup M(\aleph), & \aleph\in Ƶ\cap(B\cap Ş) \end{cases}$$

Thus,

$$N(\aleph)=\begin{cases} F'(\aleph), & \aleph\in Ƶ-(B\cap Ş)=Ƶ-(B\cap Ş) \\ G'(\aleph)\cup H'(\aleph), & \aleph\in(B\cap Ş)-Ƶ=Ƶ'\cap B\cap Ş \\ F(\aleph)\cup(G(\aleph)\cap H(\aleph)), & \aleph\in Ƶ\cap(B\cap Ş)=Ƶ\cap B\cap Ş \end{cases}$$

Now consider the RHS, that is, $[(F,Ƶ) \overset{*}{\cup}_{\varepsilon}(G,B)]\cap_R[(F,Ƶ)\overset{*}{\cup}_{\varepsilon}(H,Ş)]$. Let $(F,Ƶ)\overset{*}{\cup}_{\varepsilon}(G,B) = (V,Ƶ\cup B)$, where for all $\aleph\in Ƶ\cup B$,

$$V(ℵ)=\begin{cases} F'(ℵ), & ℵ\in Ƶ\text{-}B \\ G'(ℵ), & ℵ\in B\text{-}Ƶ \\ F(ℵ)\cup G(ℵ), & ℵ\in Ƶ\cap B \end{cases}$$

Let $(F,Ƶ)\overset{*}{\cup_\varepsilon}(H,Ş)=(W,Ƶ\cup Ş)$, where for all $ℵ\in Ƶ\cup Ş$,

$$W(ℵ)=\begin{cases} F'(ℵ), & ℵ\in Ƶ\text{-}Ş \\ H'(ℵ), & ℵ\in Ş\text{-}Ƶ \\ F(ℵ)\cup H(ℵ), & ℵ\in Ƶ\cap Ş \end{cases}$$

Let $(V,Ƶ\cup B)\cap_R(W,Ƶ\cup Ş)=(T,(Ƶ\cup B)\cap(Ƶ\cup Ş)))$, where for all $ℵ\in Ƶ\cup(B\cap Ş)$, $T(ℵ)=V(ℵ)\cap W(ℵ)$. Hence,

$$T(ℵ)=\begin{cases} F'(ℵ)\cap F'(ℵ), & ℵ\in(Ƶ\text{-}B)\cap(Ƶ\text{-}Ş)=Ƶ\cap B'\cap Ş' \\ F'(ℵ)\cap H'(ℵ), & ℵ\in(Ƶ\text{-}B)\cap(Ş\text{-}Ƶ)=\emptyset \\ F'(ℵ)\cap(F(ℵ)\cup H(ℵ)) & ℵ\in(Ƶ\text{-}B)\cap(Ƶ\cap Ş)=Ƶ\cap B'\cap Ş \\ G'(ℵ)\cap F'(ℵ), & ℵ\in(B\text{-}Ƶ)\cap(Ƶ\text{-}Ş)=\emptyset \\ G'(ℵ)\cap H'(ℵ), & ℵ\in(B\text{-}Ƶ)\cap(Ş\text{-}Ƶ)=Ƶ'\cap B\cap Ş \\ G'(ℵ)\cap(F(ℵ)\cup H(ℵ)), & ℵ\in(B\text{-}Ƶ)\cap(Ƶ\cap Ş)=\emptyset \\ (F(ℵ)\cup G(ℵ))\cap F'(ℵ), & ℵ\in(Ƶ\cap B)\cap(Ƶ\text{-}Ş)=Ƶ\cap B\cap Ş' \\ (F(ℵ)\cup G(ℵ))\cap H'(ℵ), & ℵ\in(Ƶ\cap B)\cap(Ş\text{-}Ƶ)=\emptyset \\ (F(ℵ)\cup G(ℵ))\cap(F(ℵ)\cup H(ℵ)), & ℵ\in(Ƶ\cap B)\cap(Ƶ\cap Ş)=Ƶ\cap B\cap Ş \end{cases}$$

Thus,

$$T(ℵ)=\begin{cases} F(ℵ), & ℵ\in(Ƶ\text{-}B)\cap(Ƶ\text{-}Ş)=Ƶ\cap B'\cap Ş' \\ F'(ℵ)\cap H(ℵ) & ℵ\in(Ƶ\text{-}B)\cap(Ƶ\cap Ş)=Ƶ\cap B'\cap Ş \\ G'(ℵ)\cap H'(ℵ), & ℵ\in(B\text{-}Ƶ)\cap(Ş\text{-}Ƶ)=Ƶ'\cap B\cap Ş \\ G'(ℵ)\cap F'(ℵ) & ℵ\in(Ƶ\cap B)\cap(Ƶ\text{-}Ş)=Ƶ\cap B\cap Ş' \\ F(ℵ)\cup G(ℵ))\cap(F(ℵ)\cup H(ℵ)), & ℵ\in(Ƶ\cap B)\cap(Ƶ\cap Ş)=Ƶ\cap B\cap Ş \end{cases}$$

Here, when considering the $Ƶ\text{-}(B\cap Ş)$ in the function N, since $Ƶ\text{-}(B\cap Ş)=Ƶ\text{-}(B\cap Ş)'$, if an element is in the complement of $(B\cap Ş)$, it is either in B-Ş, in Ş-B, or in $(B\cup Ş)'$. Thus, if $ℵ\in Ƶ-(B\cap Ş)$, then $ℵ\in Ƶ\cap B\cap Ş'$ or $ℵ\in Ƶ\cap B'\cap Ş$ or $ℵ\in Ƶ\cap B'\cap Ş'$. Thus, N=T under the conditions $Ƶ'\cap B\cap Ş=Ƶ\cap B'\cap Ş=Ƶ\cap B\cap Ş=Ƶ\cap B\cap Ş'=\emptyset$. It is obvious that the condition $Ƶ'\cap B\cap Ş=Ƶ\cap B'\cap Ş=\emptyset$ is equivalent to the $(Ƶ\Delta B)\cap Ş=\emptyset$.

2)If $(Ƶ\Delta B)\cap Ş=Ƶ\cap B\cap Ş'=\emptyset$, then $(F,Ƶ)\overset{*}{\cup_\varepsilon}[(G,B)\cup_R(H,Ş)]=[(F,Ƶ)\overset{*}{\cup_\varepsilon}(G,B)]\cup_R[(F,Ƶ)\overset{*}{\cup_\varepsilon}(H,Ş)]$.

3)If $(Ƶ\Delta B)\cap Ş=Ƶ\cap B\cap Ş'=\emptyset$, then $(F,Ƶ)\overset{*}{\cup_\varepsilon}[(G,B)*_R(H,Ş)]=[(F,Ƶ)\overset{*}{\lambda_\varepsilon}(G,B)\cup_R[(F,Ƶ)\overset{*}{\lambda_\varepsilon}(H,Ş)]$.

4)If $(Ƶ\Delta B)\cap Ş=Ƶ\cap B\cap Ş'=\emptyset$, then $(F,Ƶ)\overset{*}{\cup_\varepsilon}[(G,B)\theta_R(H,Ş)]=[(F,Ƶ)\overset{*}{\lambda_\varepsilon}(G,B)]\cap_R[(F,Ƶ)\overset{*}{\lambda_\varepsilon}(H,Ş)]$.

ii) RHS Distribution of Complementary Extended Union Operation over Restricted Soft Set Operations

1)If $(Ƶ\Delta B)\cap Ş=Ƶ\cap B\cap Ş'=\emptyset$, then $[(F,Ƶ)\cup_R(G,B)]\overset{*}{\cup_\varepsilon}(H,Ş)=[(F,Ƶ)\overset{*}{\cup_\varepsilon}(H,Ş)]\cup_R[(G,B)\overset{*}{\cup_\varepsilon}(H,Ş)]$.

2)If $(Ƶ\Delta B)\cap Ş=Ƶ\cap B\cap Ş'=\emptyset$, then $[(F,Ƶ)\cap_R(G,B)]\overset{*}{\cup_\varepsilon}(H,Ş)=[(F,Ƶ)\overset{*}{\cup_\varepsilon}(H,Ş)]\cap_R[(G,B)\overset{*}{\cup_\varepsilon}(H,Ş)]$.

3)If $(Ƶ\Delta B)\cap Ş=Ƶ\cap B\cap Ş'=\emptyset$, then $(F,Ƶ)*_R(G,B)]\overset{*}{\cup_\varepsilon}(H,Ş)=[(F,Ƶ)\overset{*}{+_\varepsilon}(H,Ş)]\cup_R[(G,B)\overset{*}{+_\varepsilon}(H,Ş)]$.

4)If $(Ƶ\Delta B)\cap Ş=Ƶ\cap B\cap Ş'=\emptyset$, then $(F,Ƶ)\theta_R(G,B)]\overset{*}{\cup_\varepsilon}(H,Ş)=[(F,Ƶ)\overset{*}{+_\varepsilon}(H,Ş)]\cap_R[(G,B)\overset{*}{+_\varepsilon}(H,Ş)]$.

Note 4.1.1 Considering the distributions in 4.1 and the conditions under which they are satisfied, it is obvious that the following distributions are satisfied without any conditions the set $S_Ƶ(U)$, where Ƶ is a fixed subset of the parameter set E.

- $(F,Ƶ)\overset{*}{\cup}_{\varepsilon}[(G,Ƶ)\cap_R(H,Ƶ)]=[(F,Ƶ)\overset{*}{\cup}_{\varepsilon}(G,Ƶ)]\cap_R[(F,Ƶ)\overset{*}{\cup}_{\varepsilon}(H,Ƶ)]$.
- $[(F,Ƶ)\cap_R(G,Ƶ)]\overset{*}{\cup}_{\varepsilon}(H,Ƶ)=[(F,Ƶ)\overset{*}{\cup}_{\varepsilon}(H,Ƶ)]\cap_R[(G,Ƶ)\overset{*}{\cup}_{\varepsilon}(H,Ƶ)]$.
- $(F,Ƶ)\overset{*}{\cup}_{\varepsilon}[(G,Ƶ)\cup_R(H,Ƶ)]=[(F,Ƶ)\overset{*}{\cup}_{\varepsilon}(G,Ƶ)]\cup_R[(F,Ƶ)\overset{*}{\cup}_{\varepsilon}(H,Ƶ)]$.
- $[(F,Ƶ)\cup_R(G,Ƶ)]\overset{*}{\cup}_{\varepsilon}(H,Ƶ)=[(F,Ƶ)\overset{*}{\cup}_{\varepsilon}(H,Ƶ)]\cup_R[(G,Ƶ)\overset{*}{\cup}_{\varepsilon}(H,Ƶ)]$.

Theorem 4.1.2. $(S_Ƶ(U),\cup_R,\overset{*}{\cup}_{\varepsilon})$ is a commutative, idempotent semiring without zero but with unity.
Proof: Ali et al. (2011) showed that $(S_Ƶ(U),\cup_R)$ is a commutative, idempotent monoid with identity $\emptyset_Ƶ$, thus forming a bounded semilattice (and therefore, a semigroup). By Corollary 3.3.1, $(S_Ƶ(U),\overset{*}{\cup}_{\varepsilon})$ is a commutative, idempotent monoid with identity $\emptyset_Ƶ$, that is a bounded semilattice (thus, a semigroup From Note 4.1.1, in $S_Ƶ(U)$, $\overset{*}{\cup}_{\varepsilon}$ distributes over $\cup_R$ from both sides. Thus, $(S_Ƶ(U),\cup_R,\overset{*}{\cup}_{\varepsilon})$ is a commutative, idempotent semiring without zero, but with unity.

Theorem 4.1.3 $(S_Ƶ(U),\cap_R,\overset{*}{\cup}_{\varepsilon})$ is a commutative, idempotent hemiring with unity.

Proof: Ali et al. (2011) showed that $(S_Ƶ(U),\cap_R)$ is a commutative idempotent monoid with identity $U_Ƶ$, that is a bounded semi-lattice (hence a semigroup). By Corollary 3.3.1, $(S_Ƶ(U),\overset{*}{\cup}_{\varepsilon})$ is a commutative, idempotent monoid with identity $\emptyset_Ƶ$, that is a bounded semilattice (thus, a semigroup). Also, by Note 4.1.1, in $S_Ƶ(U)$, $\overset{*}{\cup}_{\varepsilon}$ distributes over $\cap_R$ from both sides. Moreover, since $(F,Ƶ)\cap_R U_Ƶ=U_Ƶ\cap_R(F,Ƶ)=(F,Ƶ)$ and by Theorem 3.3 (7) $(F,Ƶ)\overset{*}{\cup}_{\varepsilon}U_Ƶ=U_Ƶ\overset{*}{\cup}_{\varepsilon}(F,Ƶ)=U_Ƶ$, $(S_Ƶ(U),\cap_R,\overset{*}{\cup}_{\varepsilon})$ is a commutative, idempotent hemiring with unity.

Theorem 4.1.4. $(S_Ƶ(U),U_Ƶ,\emptyset_Ƶ,\cap_R,\overset{*}{\cup}_{\varepsilon})$ Boolean Algebra and De Morgan Algebra.

Proof: Ali et al. (2011) showed that $(S_Ƶ(U),\cap_R)$ is an idempotent, commutative monoid with identity $U_Ƶ$, hence a bounded semilattice (thus, a semigroup). By Theorem 3.3.1, $(S_Ƶ(U),\overset{*}{\cup}_{\varepsilon})$ is a commutative, idempotent monoid with identity $\emptyset_Ƶ$, thus a bounded semi-lattice (hence a semigroup). By 3.3. Theorem (21), $\overset{*}{\cap}_{\varepsilon}$ and $\cap_R$ hold the absorbing law. Hence, $(S_Ƶ(U),U_Ƶ,\emptyset_Ƶ,\cap_R,\overset{*}{\cup}_{\varepsilon})$ is a bounded lattice. Moreover, since $(F,Ƶ)\cap_R(F,Ƶ)^r=\emptyset_Ƶ$ and $(F,Ƶ)\overset{*}{\cup}_{\varepsilon}(F,Ƶ)^r=U_Ƶ$ for all $(F,Ƶ)\in S_Ƶ(U)$, $(S_Ƶ(U),U_Ƶ,\emptyset_Ƶ,\cap_R,\overset{*}{\cup}_{\varepsilon})$ is a complemented bounded lattice. Furthermore, by 3.4.1.1. Corollary, $\overset{*}{\cup}_{\varepsilon}$ distributes over $\cap_R$ from both sides. Thus, $(S_Ƶ(U),U_Ƶ,\emptyset_Ƶ,\cap_R,\overset{*}{\cup}_{\varepsilon})$ is a distributive, complemented bounded lattice, hence a Boolean Algebra.

Moreover, since $[(F,Ƶ)\overset{*}{\cup}_{\varepsilon}(G,Ƶ)]^r=(F,Ƶ)^r\cap_R(G,Ƶ)^r$ and $[(F,Ƶ)\cap_R(G,Ƶ)]^r=(F,Ƶ)^r\overset{*}{\cup}_{\varepsilon}(G,Ƶ)^r$, that is, De Morgan laws hold, $(S_Ƶ(U),U_Ƶ,\emptyset_Ƶ,^c,\cap_R,\overset{*}{\cap}_{\varepsilon})$ is a De Morgan Algebra.

Here note that since $(S_E(U),\overset{*}{\cup}_{\varepsilon})$ is a non-commutative idempotent semigroup in $S_E(U)$, $\overset{*}{\cup}_{\varepsilon}$ does not form a lattice together with $\cap_R$.

Theorem 4.2. Let (F,Z), (G,B), $(H,Ş)$ be soft sets over U. Then, the following distributions of the complementary extended union operation over extended soft set operations hold:

i) LHS Distributions of the Complementary Extended Union Operation on Extended Soft Set Operations

1)If $(Z\Delta B)\cap Ş=Z\cap B\cap Ş'=\emptyset$, then $(F,Z)\overset{*}{\cup}_{\varepsilon}[(G,B)\cap_{\varepsilon}(H,Ş)]=[(F,Z)\overset{*}{\cup}_{\varepsilon}(G,B)]\cap_{\varepsilon}[(F,Z)\overset{*}{\cup}_{\varepsilon}(H,Ş)]$.

Proof: Consider first the LHS. Let $(G,B)\cap_{\varepsilon}(H,Ş)=(M,B\cup Ş)$, where for all $\aleph\in B\cup Ş$,

$$M(\aleph)=\begin{cases} G(\aleph), & \aleph\in B-Ş \\ H(\aleph), & \aleph\in Ş-B \\ G(\aleph)\cap H(\aleph), & \aleph\in B\cap Ş \end{cases}$$

Let $(F,Z)\overset{*}{\cup}_{\varepsilon}(M,B\cup Ş)=(N,Z\cup(B\cup Ş))$, where for all $\aleph\in Z\cup B\cup Ş$,

$$N(\aleph)=\begin{cases} F'(\aleph), & \aleph\in Z-(B\cup Ş) \\ M'(\aleph), & \aleph\in(B\cup Ş)-Z \\ F(\aleph)\cup M(\aleph), & \aleph\in Z\cap(B\cup Ş) \end{cases}$$

Hence,

$$N(\aleph)=\begin{cases} F'(\aleph), & \aleph\in Z-(B\cup Ş)=Z\cap B'\cap Ş' \\ G'(\aleph), & \aleph\in(B-Ş)-Z=Z'\cap B\cap Ş' \\ H'(\aleph), & \aleph\in(Ş-B)-Z=Z'\cap B'\cap Ş \\ G'(\aleph)\cup H'(\aleph), & \aleph\in(B\cap Ş)-Z=Z'\cap B\cap Ş \\ F(\aleph)\cup G(\aleph), & \aleph\in Z\cap(B-Ş)=Z\cap B\cap Ş' \\ F(\aleph)\cup H(\aleph), & \aleph\in Z\cap(Ş-B)=Z\cap B'\cap Ş \\ F(\aleph)\cup(G(\aleph)\cap H(\aleph)), & \aleph\in Z\cap(B\cap Ş)=Z\cap B\cap Ş \end{cases}$$

Now consider the RHS, i.e., $[(F,Z)\overset{*}{\cup}_{\varepsilon}(G,B)]\cap_{\varepsilon}[(F,Z)\overset{*}{\cup}_{\varepsilon}(H,Ş)]$. Let $(F,Z)\overset{*}{\cup}_{\varepsilon}(G,B)=(V,Z\cup B)$, where for all $\aleph\in Z\cup B$,

$$V(\aleph)=\begin{cases} F'(\aleph), & \aleph\in Z-B \\ G'(\aleph), & \aleph\in B-Z \\ F(\aleph)\cup G(\aleph), & \aleph\in Z\cap B \end{cases}$$

Let $(F,Z)\overset{*}{\cup}_{\varepsilon}(H,Ş)=(W,Z\cup Ş)$, where for all $\aleph\in Z\cup Ş$,

$$W(\aleph)=\begin{cases} F'(\aleph), & \aleph\in Z-Ş \\ H'(\aleph), & \aleph\in Ş-Z \\ F(\aleph)\cup H(\aleph), & \aleph\in Z\cap Ş \end{cases}$$

Let $(V,Z\cup B)\cap_{\varepsilon}(W,Z\cup Ş)=(T,(Z\cup B)\cup Ş)$, where for all $\aleph\in Z\cup B\cup Ş$,

$$T(\aleph)=\begin{cases} V(\aleph), & \aleph\in(Z\cup B)-(Z\cup Ş) \\ W(\aleph), & \aleph\in(Z\cup Ş)-(Z\cup B) \\ V(\aleph)\cap W(\aleph), & \aleph\in(Z\cup B)\cap(Z\cup Ş) \end{cases}$$

Hence,

$$
T(\aleph)=\begin{cases}
F'(\aleph), & \aleph\in(Ƶ\text{-}B)\text{-}(Ƶ\cup Ş)=\emptyset\\
G'(\aleph), & \aleph\in(B\text{-}Ƶ)\text{-}(Ƶ\cup Ş)=Ƶ'\cap B\cap Ş'\\
F(\aleph)\cup G(\aleph), & \aleph\in(Ƶ\cap B)\text{-}(Ƶ\cup Ş)=\emptyset\\
F'(\aleph), & \aleph\in(Ƶ\text{-}Ş)\text{-}(Ƶ\cup B)=\emptyset\\
H'(\aleph), & \aleph\in(Ş\text{-}Ƶ)\text{-}(Ƶ\cup B)=Ƶ'\cap B'\cap Ş\\
F(\aleph)\cup H(\aleph), & \aleph\in(Ƶ\cap Ş)\text{-}(Ƶ\cup B)=\emptyset\\
F'(\aleph)\cap F'(\aleph), & \aleph\in(Ƶ\text{-}B)\cap(Ƶ\text{-}Ş)=Ƶ\cap B'\cap Ş'\\
F'(\aleph)\cap H'(\aleph), & \aleph\in(Ƶ\text{-}B)\cap(Ş\text{-}Ƶ)=\emptyset\\
F'(\aleph)\cap(F(\aleph)\cup H(\aleph)), & \aleph\in(Ƶ\text{-}B)\cap(Ƶ\cap Ş)=Ƶ\cap B'\cap Ş\\
G'(\aleph)\cap F'(\aleph), & \aleph\in(B\text{-}Ƶ)\cap(Ƶ\text{-}Ş)=\emptyset\\
G'(\aleph)\cap H'(\aleph), & \aleph\in(B\text{-}Ƶ)\cap(Ş\text{-}Ƶ)=Ƶ'\cap B\cap Ş\\
G'(\aleph)\cap(F(\aleph)\cup H(\aleph)), & \aleph\in(B\text{-}Ƶ)\cap(Ƶ\cap Ş)=\emptyset\\
(F(\aleph)\cup G(\aleph))\cap F'(\aleph), & \aleph\in(Ƶ\cap B)\cap(Ƶ\text{-}Ş)=Ƶ\cap B\cap C'\\
(F(\aleph)\cup G(\aleph))\cap H'(\aleph), & \aleph\in(Ƶ\cap B)\cap(Ş\text{-}Ƶ)=\emptyset\\
(F(\aleph)\cup G(\aleph))\cap F(\aleph)\cup H(\aleph)), & \aleph\in(Ƶ\cap B)\cap(Ƶ\cap Ş)=Ƶ\cap B\cap Ş
\end{cases}
$$

Thus,

$$
T(\aleph)=\begin{cases}
G'(\aleph), & \aleph\in(B\text{-}Ƶ)\text{-}(Ƶ\cup Ş)=Ƶ'\cap B\cap Ş'\\
H'(\aleph), & \aleph\in(Ş\text{-}Ƶ)\text{-}(Ƶ\cup B)=Ƶ'\cap B'\cap Ş\\
F'(\aleph), & \aleph\in(Ƶ\text{-}B)\cap(Ƶ\text{-}Ş)=Ƶ\cap B'\cap Ş'\\
F'(\aleph)\cap H(\aleph), & \aleph\in(Ƶ\text{-}B)\cap(Ƶ\cap Ş)=Ƶ\cap B'\cap Ş\\
G'(\aleph)\cap H'(\aleph), & \aleph\in(B\text{-}Ƶ)\cap(Ş\text{-}Ƶ)=Ƶ'\cap B\cap Ş\\
G(\aleph)\cap H'(\aleph), & \aleph\in(Ƶ\cap B)\cap(Ƶ\text{-}Ş)=Ƶ\cap B\cap Ş'\\
(F(\aleph)\cup G(\aleph))\cap F(\aleph)\cup H(\aleph), & \aleph\in(Ƶ\cap B)\cap(Ƶ\cap Ş)=Ƶ\cap B\cap Ş
\end{cases}
$$

N=T if $Ƶ'\cap B\cap Ş=Ƶ\cap B'\cap Ş=Ƶ\cap B\cap Ş'=\emptyset$. It is obvious that the condition $Ƶ'\cap B\cap Ş=Ƶ\cap B'\cap Ş'=\emptyset$ is equivalent to the condition $(Ƶ\Delta B)\cap Ş=\emptyset$.

2)If $(Ƶ\Delta B)\cap Ş=Ƶ\cap B\cap Ş'=\emptyset$, then $(F,Ƶ)\overset{*}{\cup_\varepsilon}[(G,B)\cup_\varepsilon(H,Ş)]=[(F,Ƶ)\overset{*}{\cup_\varepsilon}(G,B)]\cup_\varepsilon[(F,Ƶ)\overset{*}{\cup_\varepsilon}(H,Ş)]$.

3)If $(Ƶ\Delta B)\cap Ş=Ƶ\cap B\cap Ş'=\emptyset$, then $(F,Ƶ)\overset{*}{\cup_\varepsilon}[(G,B)*_\varepsilon(H,Ş)]=[(F,Ƶ)\overset{*}{\lambda_\varepsilon}(G,B)]\cup_\varepsilon[(F,Ƶ)\overset{*}{\lambda_\varepsilon}(H,Ş)]$.

4)$(Ƶ\Delta B)\cap Ş=Ƶ\cap B\cap Ş'=\emptyset$, then $(F,Ƶ)\overset{*}{\cup_\varepsilon}[(G,B)\theta_\varepsilon(H,Ş)]=[(F,Ƶ)\overset{*}{\lambda_\varepsilon}(G,B)]\cap_\varepsilon[(F,Ƶ)\overset{*}{\lambda_\varepsilon}(H,Ş)]$.

ii) RHS Distributions of Complementary Extended Union Operation over Extended Soft Set Operations

1)If $(Ƶ\Delta B)\cap Ş=Ƶ\cap B\cap Ş'=\emptyset$, then $([(F,Ƶ)\cup_\varepsilon(G,B)]\overset{*}{\cup_\varepsilon}(H,Ş)=[(F,Ƶ)\overset{*}{\cup_\varepsilon}(H,Ş)]\cup_\varepsilon[(G,B)\overset{*}{\cup_\varepsilon}(H,Ş)]$.

2)If $(Ƶ\Delta B)\cap Ş=Ƶ\cap B\cap Ş'=\emptyset$, then $[(F,Ƶ)\cap_\varepsilon(G,B)]\overset{*}{\cup_\varepsilon}(H,Ş)=[(F,Ƶ)\overset{*}{\cup_\varepsilon}(H,Ş)]\cap_\varepsilon[(G,B)\overset{*}{\cup_\varepsilon}(H,Ş)]$.

3)If $(Ƶ\Delta B)\cap Ş=Ƶ\cap B\cap Ş'=\emptyset$, then $[(F,Ƶ)\theta_\varepsilon(G,B)]\overset{*}{\cup_\varepsilon}(H,Ş)=[(F,Ƶ)\overset{*}{+_\varepsilon}(H,Ş)]\cap_\varepsilon[(G,B)\overset{*}{+_\varepsilon}(H,Ş)]$.

4) If $(Ƶ\Delta B)\cap Ş=Ƶ\cap B\cap Ş'=\emptyset$, then $[(F,Ƶ)*_\varepsilon(G,B)]\overset{*}{\cup_\varepsilon}(H,Ş)=[(F,Ƶ)\overset{*}{+_\varepsilon}(H,Ş)]\cup_\varepsilon[(G,B)\overset{*}{+_\varepsilon}(H,Ş)]$.

Note 4.2.1 Considering the distributions in Theorem 4.2. and the conditions under which they are satisfied, it is obvious that the following distributions are satisfied in the set $S_Ƶ(U)$ without any conditions, where Ƶ is a fixed subset of the parameter set E:

- $(F,Ƶ)\overset{*}{\cup_\varepsilon}[(G,Ƶ)\cap_\varepsilon(H,Ƶ)]=[(F,Ƶ)\overset{*}{\cup_\varepsilon}(G,Ƶ)]\cap_\varepsilon[(F,Ƶ)\overset{*}{\cup_\varepsilon}(H,Ƶ)]$.

- [(F, Ƶ) $\cap_{\varepsilon}$(G,Ƶ)]$\overset{*}{\cup}_{\varepsilon}$(H,Ƶ)=[(F, Ƶ)$\overset{*}{\cup}_{\varepsilon}$ (H,Ƶ)]$\cap_{\varepsilon}$[(G, Ƶ)$\overset{*}{\cup}_{\varepsilon}$ (H,Ƶ)].
- (F, Ƶ)$\overset{*}{\cup}_{\varepsilon}$ [(G, Ƶ) $\cup_{\varepsilon}$(H,Ƶ)]=[(F, Ƶ)$\overset{*}{\cup}_{\varepsilon}$ (G,Ƶ)]$\cup_{\varepsilon}$[(F, Ƶ)$\overset{*}{\cup}_{\varepsilon}$(H,Ƶ)].
- [(F, Ƶ)$\cup_{\varepsilon}$(G,Ƶ)]$\overset{*}{\cup}_{\varepsilon}$(H,Ƶ)=[(F, Ƶ)$\overset{*}{\cup}_{\varepsilon}$ (H,Ƶ)]$\cup_{\varepsilon}$[(G, Ƶ)$\overset{*}{\cup}_{\varepsilon}$ (H,Ƶ)].

Theorem 4.2.2. ($S_Ƶ$(U),$\cup_{\varepsilon}$, $\overset{*}{\cup}_{\varepsilon}$) is a commutative, idempotent semiring without zero but with unity.

Theorem 4.2.3 ($S_Ƶ$(U), $\cap_{\varepsilon}$, $\overset{*}{\cup}_{\varepsilon}$) is a commutative, idempotent hemiring with unity.

Theorem 4.2.4. ($S_Ƶ$(U),$U_Ƶ$, $\emptyset_Ƶ$, $\cap_{\varepsilon}$, $\overset{*}{\cup}_{\varepsilon}$) Bool Algebra and De Morgan Algebra.

Theorem 4.3. Let (F, Ƶ), (G, B), (H,Ş) be soft sets over U. The following distributions of the complementary extended union operation over complementary extended operations hold:

i) LHS Distributions of Complementary Extended Union Operations over Complementary Extended Soft Set Operations

1)If (Ƶ∆B)∩ Ş=Ƶ∩B∩Ş'=∅, then (F, Ƶ)$\overset{*}{\cup}_{\varepsilon}$[(G,B) $\overset{*}{\cap}_{\varepsilon}$(H,Ş)]=[(F, Ƶ)$\overset{*}{\cup}_{\varepsilon}$ (G,B)] $\overset{*}{\cap}_{\varepsilon}$[(F, Ƶ)$\overset{*}{\cup}_{\varepsilon}$ (H,Ş)].

Proof: Consider first the LHS. Let (G,B) $\overset{*}{\cap}_{\varepsilon}$(H,Ş)=(M,B∪Ş), where for all ℵ∈B∪Ş,

$$M(\aleph)=\begin{cases} G'(\aleph), & \aleph\in B-Ş \\ H'(\aleph), & \aleph\in Ş-B \\ G(\aleph)\cap H(\aleph), & \aleph\in B\cap Ş \end{cases}$$

Let (F, Ƶ)$\overset{*}{\cup}_{\varepsilon}$ (M,B∪Ş)=(N,Ƶ∪(B∪Ş)), where for all ℵ∈Ƶ∪B∪Ş,

$$N(\aleph)=\begin{cases} F'(\aleph), & \aleph\in Ƶ-(B\cup Ş) \\ M'(\aleph), & \aleph\in (B\cup Ş)-Ƶ \\ F(\aleph)\cup M(\aleph), & \aleph\in Ƶ\cap(B\cup Ş) \end{cases}$$

Thus,

$$N(\aleph)=\begin{cases} F'(\aleph), & \aleph\in Ƶ-(B\cup Ş)=Ƶ\cap B'\cap Ş' \\ G(\aleph), & \aleph\in (B-Ş)-Ƶ=Ƶ'\cap B\cap Ş' \\ H(\aleph), & \aleph\in (Ş-B)-Ƶ=Ƶ'\cap B'\cap Ş \\ G'(\aleph)\cup H'(\aleph), & \aleph\in (B\cap Ş)-Ƶ=Ƶ'\cap B\cap Ş \\ F(\aleph)\cup G(\aleph), & \aleph\in Ƶ\cap(B-Ş)=Ƶ\cap B\cap Ş' \\ F(\aleph)\cup H(\aleph), & \aleph\in Ƶ\cap(Ş-B)=Ƶ\cap B'\cap Ş \\ F(\aleph)\cup(G(\aleph)\cap H(\aleph)), & \aleph\in Ƶ\cap(B\cap Ş)=Ƶ\cap B\cap Ş \end{cases}$$

Now consider the RHS, i.e., [(F, Ƶ)$\overset{*}{\cup}_{\varepsilon}$ (G,B)] $\overset{*}{\cap}_{\varepsilon}$[(F, Ƶ)$\overset{*}{\cup}_{\varepsilon}$ (H,Ş)]. Let (F, Ƶ)$\overset{*}{\cup}_{\varepsilon}$ (G,B) = (V,Ƶ∪B), where for all ℵ∈Ƶ∪B,

$$V(\aleph)=\begin{cases} F'(\aleph), & \aleph\in Ƶ-B \\ G'(\aleph), & \aleph\in B-Ƶ \\ F(\aleph)\cup G(\aleph), & \aleph\in Ƶ\cap B \end{cases}$$

Let (F,Ƶ) $\overset{*}{\cup}_{\varepsilon}$ (H,Ş)=(W,Ƶ∪Ş), where for all ℵ∈Ƶ∪Ş,

$$W(\aleph)=\begin{cases} F'(\aleph), & \aleph\in Ƶ-Ş \\ H'(\aleph), & \aleph\in Ş-Ƶ \\ F(\aleph)\cup H(\aleph), & \aleph\in Ƶ\cap Ş \end{cases}$$

Let $(V,Ƶ\cup B)\overset{*}{\cap}_{\varepsilon}(W,Ƶ\cup Ş)=(T,(Ƶ\cup B)\cup Ş)$, where for all $\aleph\in Ƶ\cup B\cup Ş$,

$$T(\aleph)=\begin{cases} V'(\aleph), & \aleph\in(Ƶ\cup B)-(Ƶ\cup Ş) \\ W'(\aleph), & \aleph\in(Ƶ\cup Ş)-(Ƶ\cup B) \\ V(\aleph)\cap W(\aleph), & \aleph\in(Ƶ\cup B)\cap(Ƶ\cup Ş) \end{cases}$$

Hence,

$$T(\aleph)=\begin{cases} F(\aleph), & \aleph\in(Ƶ-B)-(Ƶ\cup Ş)=\emptyset \\ G(\aleph), & \aleph\in(B-Ƶ)-(Ƶ\cup Ş)=Ƶ'\cap B\cap Ş' \\ F'(\aleph)\cap G'(\aleph), & \aleph\in(Ƶ\cap B)-(Ƶ\cup Ş)=\emptyset \\ F(\aleph), & \aleph\in(Ƶ-Ş)-(Ƶ\cup B)=\emptyset \\ H(\aleph), & \aleph\in(Ş-Ƶ)-(Ƶ\cup B)=Ƶ'\cap B'\cap Ş \\ F'(\aleph)\cap H'(\aleph), & \aleph\in(Ƶ\cap Ş)-(Ƶ\cup B)=\emptyset \\ F'(\aleph)\cap F'(\aleph), & \aleph\in(Ƶ-B)\cap(Ƶ-Ş)=Ƶ\cap B'\cap Ş' \\ F'(\aleph)\cap H'(\aleph), & \aleph\in(Ƶ-B)\cap(Ş-Ƶ)=\emptyset \\ F'(\aleph)\cap(F(\aleph)\cup H(\aleph)), & \aleph\in(Ƶ-B)\cap(Ƶ\cap Ş)=Ƶ\cap B'\cap Ş \\ G'(\aleph)\cap F'(\aleph), & \aleph\in(B-Ƶ)\cap(Ƶ-Ş)=\emptyset \\ G'(\aleph)\cap H'(\aleph), & \aleph\in(B-Ƶ)\cap(Ş-Ƶ)=Ƶ'\cap B\cap Ş \\ G'(\aleph)\cap(F(\aleph)\cup H(\aleph)), & \aleph\in(B-Ƶ)\cap(Ƶ\cap Ş)=\emptyset \\ (F(\aleph)\cup G(\aleph))\cap F'(\aleph), & \aleph\in(Ƶ\cap B)\cap(Ƶ-Ş)=Ƶ\cap B\cap Ş' \\ (F(\aleph)\cup G(\aleph))\cap H'(\aleph), & \aleph\in(Ƶ\cap B)\cap(Ş-Ƶ)=\emptyset \\ (F(\aleph)\cup G(\aleph))\cap(F(\aleph)\cup H(\aleph)), & \aleph\in(Ƶ\cap B)\cap(Ƶ\cap Ş)=Ƶ\cap B\cap Ş \end{cases}$$

Therefore,

$$T(\aleph)=\begin{cases} G(\aleph), & \aleph\in(B-Ƶ)-(Ƶ\cup Ş)=Ƶ'\cap B\cap Ş' \\ H(\aleph), & \aleph\in(Ş-Ƶ)-(Ƶ\cup B)=Ƶ'\cap B'\cap Ş \\ F'(\aleph) & \aleph\in(Ƶ-B)\cap(Ƶ-Ş)=Ƶ\cap B'\cap Ş' \\ F'(\aleph)\cap H(\aleph) & \aleph\in(Ƶ-B)\cap(Ƶ\cap Ş)=Ƶ\cap B'\cap Ş \\ G'(\aleph)\cap H'(\aleph), & \aleph\in(B-Ƶ)\cap(Ş-Ƶ)=Ƶ'\cap B\cap Ş \\ F'(\aleph)\cap F'(\aleph) & \aleph\in(Ƶ\cap B)\cap(Ƶ-Ş)=Ƶ\cap B\cap Ş' \\ (F(\aleph)\cup G(\aleph))\cap(F(\aleph)\cup H(\aleph)), & \aleph\in(Ƶ\cap B)\cap(Ƶ\cap Ş)=Ƶ\cap B\cap Ş \end{cases}$$

N=T is satisfied under $Ƶ'\cap B\cap Ş=Ƶ\cap B'\cap Ş=Ƶ\cap B\cap Ş'=\emptyset$. It is obvious that the condition $Ƶ'\cap B\cap Ş=Ƶ\cap B'\cap Ş=\emptyset$ is equivalent to the condition $(Ƶ\Delta B)\cap Ş=\emptyset$.

2)If $(Ƶ\Delta B)\cap Ş=Ƶ\cap B\cap Ş'=\emptyset$, then $(F,Ƶ)\overset{*}{\cup}_{\varepsilon}[(G,B)\overset{*}{\cup}_{\varepsilon}(H,Ş)]=[(F,Ƶ)\overset{*}{\cup}_{\varepsilon}(G,B)]\overset{*}{\cup}_{\varepsilon}[(F,Ƶ)\overset{*}{\cup}_{\varepsilon}(H,Ş)]$.

3)If $(Ƶ\Delta B)\cap Ş=Ƶ\cap B\cap Ş'=\emptyset$, then $(F,Ƶ)\overset{*}{\cup}_{\varepsilon}[(G,B)\overset{*}{*}_{\varepsilon}(H,Ş)]=[(F,Ƶ)\overset{*}{\lambda}_{\varepsilon}(G,B)]\overset{*}{\cup}_{\varepsilon}[(F,Ƶ)\overset{*}{\lambda}_{\varepsilon}(H,Ş)]$.

4) If $(Ƶ\Delta B)\cap Ş=Ƶ\cap B\cap Ş'=\emptyset$, then $(F,Ƶ)\overset{*}{\cup}_{\varepsilon}[(G,B)\overset{*}{\theta}_{\varepsilon}(H,Ş)]=[(F,Ƶ)\overset{*}{\lambda}_{\varepsilon}(G,B)]\overset{*}{\cap}_{\varepsilon}[(F,Ƶ)\overset{*}{\lambda}_{\varepsilon}(H,Ş)]$.

ii) RHS Distributions of Complementary Extended Union Operations over Complementary Extended Soft Set Operations

1)If $(Ƶ\Delta B)\cap Ş=Ƶ\cap B\cap Ş'=\emptyset$, then $[(F,Ƶ)\overset{*}{\cup}_{\varepsilon}(G,B)]\overset{*}{\cup}_{\varepsilon}(H,Ş)=[(F,Ƶ)\overset{*}{\cup}_{\varepsilon}(H,Ş)]\overset{*}{\cup}_{\varepsilon}[(G,B)\overset{*}{\cup}_{\varepsilon}(H,Ş)]$.

2)If $(Ƶ\Delta B)\cap Ş=Ƶ\cap B\cap Ş'=\emptyset$, then $[(F,Ƶ)\overset{*}{\cap}_{\varepsilon}(G,B)]\overset{*}{\cup}_{\varepsilon}(H,Ş)=[(F,Ƶ)\overset{*}{\cup}_{\varepsilon}(H,Ş)]\overset{*}{\cap}_{\varepsilon}[(G,B)\overset{*}{\cup}_{\varepsilon}(H,Ş)]$.

3)If $(Ƶ\Delta B)\cap Ş=Ƶ\cap B\cap Ş'=\emptyset$, then $[(F,Ƶ)\overset{*}{\theta}_{\varepsilon}(G,B)]\overset{*}{\cup}_{\varepsilon}(H,Ş)=[(F,Ƶ)\overset{*}{\lambda}_{\varepsilon}(H,Ş)]\overset{*}{\cap}_{\varepsilon}[(G,B)\overset{*}{\lambda}_{\varepsilon}(H,Ş)]$.

4)If $(Ƶ\Delta B)\cap Ş=Ƶ\cap B\cap Ş'=\emptyset$, then $[(F,Ƶ)\overset{*}{*}_{\varepsilon}(G,B)]\overset{*}{\cup}_{\varepsilon}(H,Ş)=[(F,Ƶ)\overset{*}{+}_{\varepsilon}(H,Ş)]\overset{*}{\cup}_{\varepsilon}[(G,B)\overset{*}{+}_{\varepsilon}(H,Ş)]$.

Note 4.3.1 If we consider the distributions in Theorem 4.3 and the conditions under which they are satisfied, it is obvious that the following distributions are satisfied without any conditions in the set $S_Ƶ(U)$, where Ƶ is a fixed subset of the parameter set E.

- $(F,Ƶ)\overset{*}{\cup}_{\varepsilon}[(G,Ƶ)\overset{*}{\cap}_{\varepsilon}(H,Ƶ)]=[(F,Ƶ)\overset{*}{\cup}_{\varepsilon}(G,Ƶ)]\overset{*}{\cap}_{\varepsilon}[(F,Ƶ)\overset{*}{\cup}_{\varepsilon}(H,Ƶ)]$.
- $[(F,Ƶ)\overset{*}{\cap}_{\varepsilon}(G,Ƶ)]\overset{*}{\cup}_{\varepsilon}(H,Ƶ)=[(F,Ƶ)\overset{*}{\cup}_{\varepsilon}(H,Ƶ)]\overset{*}{\cap}_{\varepsilon}[(G,Ƶ)\overset{*}{\cup}_{\varepsilon}(H,Ƶ)]$.
- $(F,Ƶ)\overset{*}{\cup}_{\varepsilon}[(G,Ƶ)\overset{*}{\cup}_{\varepsilon}(H,Ƶ)]=[(F,Ƶ)\overset{*}{\cup}_{\varepsilon}(G,Ƶ)]\overset{*}{\cup}_{\varepsilon}[(F,Ƶ)\overset{*}{\cup}_{\varepsilon}(H,Ƶ)]$.
- $[(F,Ƶ)\overset{*}{\cup}_{\varepsilon}(G,Ƶ)]\overset{*}{\cup}_{\varepsilon}(H,Ƶ)=[(F,Ƶ)\overset{*}{\cup}_{\varepsilon}(H,Ƶ)]\overset{*}{\cup}_{\varepsilon}[(G,Ƶ)\overset{*}{\cup}_{\varepsilon}(H,Ƶ)]$.

Theorem 4.3.2. $(S_Ƶ(U),\overset{*}{\cup}_{\varepsilon},\overset{*}{\cup}_{\varepsilon})$ is a commutative, idempotent semiring without zero but with unity.

Theorem 4.3.3 $(S_Ƶ(U),\overset{*}{\cap}_{\varepsilon},\overset{*}{\cup}_{\varepsilon})$ is a commutative, idempotent hemiring with unity.

Theorem 4.3.4. $(S_Ƶ(U),U_Ƶ,\emptyset_Ƶ,\overset{*}{\cap}_{\varepsilon},\overset{*}{\cup}_{\varepsilon})$ Bool Algebra and De Morgan Algebra.

Theorem 4.4. Let $(F,Ƶ),(G,B),(H,Ş)$ be soft sets over U. The following distributions of the complementary extended union operation over soft binary piecewise operations hold:

i) LHS Distributions of Complementary Extended Union Operations over Soft Binary Piecewise Soft Set Operations

1)$(Ƶ\Delta B)\cap Ş=Ƶ\cap B\cap Ş'=\emptyset$ if $(F,Ƶ)\overset{*}{\cup}_{\varepsilon}[(G,B)\tilde{\cap}(H,Ş)]=[(F,Ƶ)\overset{*}{\cup}_{\varepsilon}(G,B)]\tilde{\cap}[(F,Ƶ)\overset{*}{\cup}_{\varepsilon}(H,Ş)]$.

Proof: Consider first the LHS. Let $(G,B)\tilde{\cap}(H,Ş)=(M,B)$, where for all $\aleph\in B$,

$$M(\aleph)=\begin{cases} G(\aleph), & \aleph\in B-Ş \\ G(\aleph)\cap H(\aleph), & \aleph\in B\cap Ş \end{cases}$$

Let $(F,Ƶ)\overset{*}{\cup}_{\varepsilon}(M,B)=(N,Ƶ\cup B)$, where for all $\aleph\in Ƶ\cup B$,

$$N(\aleph)=\begin{cases} F'(\aleph), & \aleph\in Ƶ-B \\ M'(\aleph), & \aleph\in B-Ƶ \\ F(\aleph)\cup M(\aleph), & \aleph\in Ƶ\cap B \end{cases}$$

Hence,

$$N(\aleph)=\begin{cases} F'(\aleph), & \aleph\in Ƶ-B \\ G'(\aleph), & \aleph\in(B-Ş)-Ƶ=Ƶ'\cap B\cap Ş' \\ G'(\aleph)\cup H'(\aleph), & \aleph\in(B\cap Ş)-Ƶ=Ƶ'\cap B\cap Ş \\ F(\aleph)\cup G(\aleph), & \aleph\in Ƶ\cap(B-Ş)=Ƶ\cap B\cap Ş' \\ F(\aleph)\cup(G(\aleph)\cap H(\aleph)), & \aleph\in Ƶ\cap(B\cap Ş)=Ƶ\cap B\cap Ş \end{cases}$$

Now consider the RHS, i.e., $[(F,Ƶ)\overset{*}{\cup}_{\varepsilon}(G,B)]\tilde{\cap}[(F,Ƶ)\overset{*}{\cup}_{\varepsilon}(H,Ş)]$. Let $(F,Ƶ)\overset{*}{\cup}_{\varepsilon}(G,B)=(V,Ƶ\cup B)$, where for all $\aleph\in Ƶ\cup B$,

$$V(\aleph)=\begin{cases} F'(\aleph), & \aleph\in Ƶ-B \\ G'(\aleph), & \aleph\in B-Ƶ \\ F(\aleph)\cup G(\aleph), & \aleph\in Ƶ\cap B \end{cases}$$

Let $(F,Ƶ)\overset{*}{\cup}_{\varepsilon}(H,Ş)=(W,Ƶ\cup Ş)$, where for all $ℵ\in Ƶ\cup Ş$,

$$W(ℵ)=\begin{cases} F'(ℵ), & ℵ\in Ƶ-Ş \\ H'(ℵ), & ℵ\in Ş-Ƶ \\ F(ℵ)\cup H(ℵ), & ℵ\in Ƶ\cap Ş \end{cases}$$

Let $(V,Ƶ\cup B)\tilde{\cap}(W,Ƶ\cup Ş)=(T,(Ƶ\cup B))$, where for all $ℵ\in Ƶ\cup B$,

$$T(ℵ)=\begin{cases} V(ℵ), & ℵ\in(Ƶ\cup B)-(Ƶ\cup Ş) \\ V(ℵ)\cap W(ℵ), & ℵ\in(Ƶ\cup B)\cap(Ƶ\cup Ş) \end{cases}$$

Hence,

$$T(ℵ)=\begin{cases} F'(ℵ), & ℵ\in(Ƶ\text{-}B)-(Ƶ\cup Ş)=\emptyset \\ G'(ℵ), & ℵ\in(B\text{-}Ƶ)-(Ƶ\cup Ş)=Ƶ'\cap B\cap Ş' \\ F(ℵ)\cup G(ℵ), & ℵ\in(Ƶ\cap B)-(Ƶ\cup Ş)=\emptyset \\ F'(ℵ)\cap F'(ℵ), & ℵ\in(Ƶ\text{-}B)\cap(Ƶ\text{-}Ş)=Ƶ\cap B'\cap Ş' \\ F'(ℵ)\cap H'(ℵ), & ℵ\in(Ƶ\text{-}B)\cap(Ş\text{-}Ƶ)=\emptyset \\ F'(ℵ)\cap(F(ℵ)\cup H(ℵ)), & ℵ\in(Ƶ\text{-}B)\cap(Ƶ\cap Ş)=Ƶ\cap B'\cap Ş \\ G'(ℵ)\cap F'(ℵ), & ℵ\in(B\text{-}Ƶ)\cap(Ƶ\text{-}Ş)=\emptyset \\ G'(ℵ)\cap H'(ℵ), & ℵ\in(B\text{-}Ƶ)\cap(Ş\text{-}Ƶ)=Ƶ'\cap B\cap Ş \\ G'(ℵ)\cap(F(ℵ)\cup H(ℵ)), & ℵ\in(B\text{-}Ƶ)\cap(Ƶ\cap Ş)=\emptyset \\ (F(ℵ)\cup G(ℵ))\cap F'(ℵ), & ℵ\in(Ƶ\cap B)\cap(Ƶ\text{-}Ş)=Ƶ\cap B\cap Ş' \\ (F(ℵ)\cup G(ℵ))\cap H'(ℵ), & ℵ\in(Ƶ\cap B)\cap(Ş\text{-}Ƶ)=\emptyset \\ (F(ℵ)\cup G(ℵ))\cap(F(ℵ)\cup H(ℵ)), & ℵ\in(Ƶ\cap B)\cap(Ƶ\cap Ş)=Ƶ\cap B\cap Ş \end{cases}$$

Hence,

$$T(ℵ)=\begin{cases} G'(ℵ), & ℵ\in(B\text{-}Ƶ)-(Ƶ\cup Ş)=Ƶ'\cap B\cap Ş' \\ F'(ℵ), & ℵ\in(Ƶ\text{-}B)\cap(Ƶ\text{-}Ş)=Ƶ\cap B'\cap Ş' \\ F'(ℵ)\cap H(ℵ), & ℵ\in(Ƶ\text{-}B)\cap(Ƶ\cap Ş)=Ƶ\cap B'\cap Ş \\ G'(ℵ)\cap H'(ℵ), & ℵ\in(B\text{-}Ƶ)\cap(Ş\text{-}Ƶ)=Ƶ'\cap B\cap Ş \\ G(ℵ)\cap H'(ℵ) & ℵ\in(Ƶ\cap B)\cap(Ƶ\text{-}Ş)=Ƶ\cap B\cap Ş' \\ (F(ℵ)\cup G(ℵ))\cap(F(ℵ)\cup H(ℵ)), & ℵ\in(Ƶ\cap B)\cap(Ƶ\cap Ş)=Ƶ\cap B\cap Ş \end{cases}$$

Here, if we consider Ƶ-B in the function N, since $Ƶ\text{-}B=Ƶ\cap B'$, if an element is in the complement of B, then the element is either in Ş-B or in $(B\cup Ş)'$. If $ℵ\in Ƶ\text{-}B$, then $ℵ\in Ƶ\cap B'\cap Ş$ or $ℵ\in Ƶ\cap B'\cap Ş'$. N=T under the $Ƶ'\cap B\cap Ş=Ƶ\cap B'\cap Ş=Ƶ\cap B\cap Ş'=\emptyset$. It is obvious that the condition $Ƶ'\cap B\cap Ş=Ƶ\cap B'\cap Ş=\emptyset$ is equivalent to the condition $(Ƶ\Delta B)\cap Ş$.

2)If $(Ƶ\Delta B)\cap Ş=Ƶ\cap B\cap Ş'=\emptyset$, then $(F,Ƶ)\overset{*}{\cup}_{\varepsilon}[(G,B)\tilde{\cup}(H,Ş)]=[(F,Ƶ)\overset{*}{\cup}_{\varepsilon}(G,B)]\tilde{\cup}[(F,Ƶ)\overset{*}{\cup}_{\varepsilon}(H,Ş)]$.

3)If $(Ƶ\Delta B)\cap C=Ƶ\cap B\cap Ş'=\emptyset$, then $(F,Ƶ)\overset{*}{\cup}_{\varepsilon}[(G,B\,\tilde{*}\,(H,Ş)]=[(F,Ƶ)\overset{*}{\lambda}_{\varepsilon}(G,B)]\tilde{\cup}[(F,Ƶ)\overset{*}{\lambda}_{\varepsilon}(H,Ş)]$.

4) If $(Ƶ\Delta B)\cap Ş=Ƶ\cap B\cap Ş'=\emptyset$, then $(F,Ƶ)\overset{*}{\cap}_{\varepsilon}[(G,B)\tilde{\theta}(H,Ş)]=[(F,Ƶ)\overset{*}{\lambda}_{\varepsilon}(G,B)]\tilde{\cap}[(F,Ƶ)\overset{*}{\lambda}_{\varepsilon}(H,Ş)]$.

ii) RHS Distributions of Complementary Extended Union Operations over Soft Binary Piecewise Soft Set Operations

1) If $(Ƶ\Delta B)\cap Ş=Ƶ\cap B\cap Ş'=\emptyset$, $(F,Ƶ)\tilde{\cup}(G,B)]\overset{*}{\cup}_{\varepsilon}(H,Ş)=[(F,Ƶ)\overset{*}{\cap}_{\varepsilon}(H,Ş)]\tilde{\cup}[(G,B)\overset{*}{\cup}_{\varepsilon}(H,Ş)]$.

2)If $(Ƶ\Delta B)\cap Ş=Ƶ\cap B\cap Ş'=\emptyset$, then $(F,Ƶ)\tilde{\cap}(G,B)]\overset{*}{\cup}_{\varepsilon}(H,Ş)=[(F,Ƶ)\overset{*}{\cup}_{\varepsilon}(H,Ş)]\tilde{\cap}[(G,B)\overset{*}{\cup}_{\varepsilon}(H,Ş)]$.

3)If $(Ƶ\Delta B)\cap Ş=Ƶ\cap B\cap Ş'=\emptyset$, then $[(F,Ƶ)\tilde{\theta}(G,B)]\overset{*}{\cup}_{\varepsilon}(H,Ş)=[(F,Ƶ)\overset{*}{+}_{\varepsilon}(H,Ş)]\tilde{\cap}[(G,B)\overset{*}{+}_{\varepsilon}(H,Ş)]$.

4)If $(Ƶ\Delta B)\cap Ş=Ƶ\cap B\cap Ş'=\emptyset$, then $[(F,Ƶ)\underset{*}{\sim}(G,B)]\overset{*}{\cap}_{\varepsilon}(H,Ş)=[(F,Ƶ)\overset{*}{+}_{\varepsilon}(H,Ş)]\tilde{\cup}[(G,B)\overset{*}{+}_{\varepsilon}(H,Ş)]$.

Note 4.4.1 If we consider the distributions in Theorem 4.4 and the conditions under which they are satisfied, it is obvious that the following distributions are satisfied in $S_{Ƶ}(U)$ without any conditions, where Ƶ is a fixed subset of the parameter set E.

- $(F,Ƶ)\overset{*}{\cup}_{\varepsilon}[(G,Ƶ)\tilde{\cap}(H,Ƶ)]=[(F,Ƶ)\overset{*}{\cup}_{\varepsilon}(G,Ƶ)]\tilde{\cap}[(F,Ƶ)\overset{*}{\cup}_{\varepsilon}(H,Ƶ)]$.
- $[(F,Ƶ)\tilde{\cap}(G,Ƶ)]\overset{*}{\cap}_{\varepsilon}(H,Ƶ)=[(F,Ƶ)\overset{*}{\cup}_{\varepsilon}(H,Ƶ)]\tilde{\cap}[(G,Ƶ)\overset{*}{\cup}_{\varepsilon}(H,Ƶ)]$.
- $(F,Ƶ)\overset{*}{\cup}_{\varepsilon}[(G,Ƶ)\tilde{\cup}(H,Ƶ)]=[(F,Ƶ)\overset{*}{\cup}_{\varepsilon}(G,Ƶ)]\tilde{\cup}[(F,Ƶ)\overset{*}{\cup}_{\varepsilon}(H,Ƶ)]$.
- $[(F,Ƶ)\tilde{\cup}(G,Ƶ)]\overset{*}{\cup}_{\varepsilon}(H,Ƶ)=[(F,Ƶ)\overset{*}{\cup}_{\varepsilon}(H,Ƶ)]\tilde{\cup}[(G,Ƶ)\overset{*}{\cup}_{\varepsilon}(H,Ƶ)]$.

Theorem 4.4.2. $(S_{Ƶ}(U),\tilde{\cup},\overset{*}{\cup}_{\varepsilon})$ is a commutative, idempotent semiring without zero but with unity.

Theorem 4.4.3 $(S_{Ƶ}(U),\tilde{\cap},\overset{*}{\cup}_{\varepsilon})$ is a commutative, idempotent hemiring with unity.

Theorem 4.4.4. $(S_{Ƶ}(U),U_{Ƶ},\emptyset_{Ƶ},\tilde{\cap},\overset{*}{\cup}_{\varepsilon})$ Bool Algebra and De Morgan Algebra.

Theorem 4.5. Let $(F,Ƶ),(G,B),(H,Ş)$ be soft sets over U. The following distributions of the complementary extended union operation over complementary soft binary piecewise operations hold:

i) LHS Distributions of Complementary Extended Union Operations over Complementary Soft Binary Piecewise Soft Set Operations

1)If $(Ƶ\Delta B)\cap Ş=Ƶ\cap B\cap Ş'=\emptyset$, then $(F,Ƶ)\overset{*}{\cup}_{\varepsilon}[(G,B)\overset{*}{\underset{\cap}{\sim}}(H,Ş)]=[(F,Ƶ)\overset{*}{\cup}_{\varepsilon}(G,B)]\overset{*}{\underset{\cap}{\sim}}[(F,Ƶ)\overset{*}{\cup}_{\varepsilon}(H,Ş)]$.

Proof: Consider first the LHS. Let $(G,B)\overset{*}{\underset{\cap}{\sim}}(H,Ş)=(M,B)$. Hence, for all $\aleph\in B$,

$$M(\aleph)=\begin{cases}G'(\aleph), & \aleph\in B-Ş\\ G(\aleph)\cap H(\aleph), & \aleph\in B\cap Ş\end{cases}$$

Let $(F,Ƶ)\overset{*}{\cup}_{\varepsilon}(M,B)=(N,Ƶ\cup B)$, where for all $\aleph\in Ƶ\cup B$,

$$N(\aleph)=\begin{cases}F'(\aleph), & \aleph\in Ƶ-B\\ M'(\aleph), & \aleph\in B-Ƶ\\ F(\aleph)\cup M(\aleph), & \aleph\in Ƶ\cap B\end{cases}$$

Hence,

$$N(\aleph)=\begin{cases}F'(\aleph), & \aleph\in Ƶ-B\\ G(\aleph), & \aleph\in(B-Ş)-Ƶ=Ƶ'\cap B\cap Ş'\\ G'(\aleph)\cup H'(\aleph), & \aleph\in(B\cap Ş)-Ƶ=Ƶ'\cap B\cap Ş\\ F(\aleph)\cup G'(\aleph), & \aleph\in Ƶ\cap(B-Ş)=Ƶ\cap B\cap Ş'\\ F(\aleph)\cup(G(\aleph)\cap H(\aleph)), & \aleph\in Ƶ\cap(B\cap Ş)=Ƶ\cap B\cap Ş\end{cases}$$

Now consider the RHS, i.e., $[(F,Ƶ)\overset{*}{\cup}_{\varepsilon}(G,B)]\overset{*}{\underset{\cap}{\sim}}[(F,Ƶ)\overset{*}{\cup}_{\varepsilon}(H,Ş)]$. Let $(F,Ƶ)\overset{*}{\cup}_{\varepsilon}(G,B)=(V,Ƶ\cup B)$, where for all $\aleph\in Ƶ\cup B$,

$$V(\aleph)=\begin{cases}F'(\aleph), & \aleph\in Ƶ-B\\ G'(\aleph), & \aleph\in B-Ƶ\\ F(\aleph)\cup G(\aleph), & \aleph\in Ƶ\cap B\end{cases}$$

Let $(F,Ƶ) \overset{*}{\cup_{\varepsilon}} (H,Ş)=(W,Ƶ\cup Ş)$, where for all $\aleph\in Ƶ\cup Ş$,

$$W(\aleph)=\begin{cases} F'(\aleph), & \aleph\in Ƶ\text{-}Ş \\ H'(\aleph), & \aleph\in Ş\text{-}Ƶ \\ F(\aleph)\cup H(\aleph), & \aleph\in Ƶ\cap Ş \end{cases}$$

Let $(V,Ƶ\cup B) \overset{*}{\underset{\cap}{\sim}} (W,Ƶ\cup Ş)=(T,(Ƶ\cup B))$, where for all $\aleph\in Ƶ\cup B$,

$$T(\aleph)=\begin{cases} V'(\aleph), & \aleph\in(Ƶ\cup B)\text{-}(Ƶ\cup Ş) \\ V(\aleph)\cap W(\aleph), & \aleph\in(Ƶ\cup B)\cap(Ƶ\cup Ş) \end{cases}$$

Hence,

$$T(\aleph)=\begin{cases} F(\aleph), & \aleph\in(Ƶ\text{-}B)\text{-}(Ƶ\cup Ş)=\emptyset \\ G(\aleph), & \aleph\in(B\text{-}Ƶ)\text{-}(Ƶ\cup Ş)=Ƶ'\cap B\cap Ş' \\ F'(\aleph)\cap G'(\aleph), & \aleph\in(Ƶ\cap B)\text{-}(Ƶ\cup Ş)=\emptyset \\ F'(\aleph)\cap F'(\aleph), & \aleph\in(Ƶ\text{-}B)\cap(Ƶ\text{-}Ş)=Ƶ\cap B'\cap Ş' \\ F'(\aleph)\cap H'(\aleph), & \aleph\in(Ƶ\text{-}B)\cap(Ş\text{-}Ƶ)=\emptyset \\ F'(\aleph)\cap(F(\aleph)\cup H(\aleph)), & \aleph\in(Ƶ\text{-}B)\cap(Ƶ\cap Ş)=Ƶ\cap B'\cap Ş \\ G'(\aleph)\cap F'(\aleph), & \aleph\in(B\text{-}Ƶ)\cap(Ƶ\text{-}Ş)=\emptyset \\ G'(\aleph)\cap H'(\aleph), & \aleph\in(B\text{-}Ƶ)\cap(Ş\text{-}Ƶ)=Ƶ'\cap B\cap Ş \\ G'(\aleph)\cap(F(\aleph)\cup H(\aleph)), & \aleph\in(B\text{-}Ƶ)\cap(Ƶ\cap Ş)=\emptyset \\ (F(\aleph)\cup G(\aleph))\cap F'(\aleph), & \aleph\in(Ƶ\cap B)\cap(Ƶ\text{-}Ş)=Ƶ\cap B\cap Ş' \\ (F(\aleph)\cup G(\aleph))\cap H'(\aleph), & \aleph\in(Ƶ\cap B)\cap(Ş\text{-}Ƶ)=\emptyset \\ ((F(\aleph)\cup G(\aleph))\cap(F(\aleph)\cup H(\aleph)), & \aleph\in(Ƶ\cap B)\cap(Ƶ\cap Ş)=Ƶ\cap B\cap Ş \end{cases}$$

Hence,

$$T(\aleph)=\begin{cases} G(\aleph), & \aleph\in(B\text{-}Ƶ)\text{-}(Ƶ\cup Ş)=Ƶ'\cap B\cap Ş' \\ F'(\aleph), & \aleph\in(Ƶ\text{-}B)\cap(Ƶ\text{-}Ş)=Ƶ\cap B'\cap Ş' \\ F'(\aleph)\cap H'(\aleph), & \aleph\in(Ƶ\text{-}B)\cap(Ƶ\cap Ş)=Ƶ\cap B'\cap Ş \\ G'(\aleph)\cap H'(\aleph), & \aleph\in(B\text{-}Ƶ)\cap(Ş\text{-}Ƶ)=Ƶ'\cap B\cap Ş \\ G(\aleph)\cap F'(\aleph), & \aleph\in(Ƶ\cap B)\cap(Ƶ\text{-}Ş)=Ƶ\cap B\cap Ş' \\ ((F(\aleph)\cup G(\aleph))\cap(F(\aleph)\cup H(\aleph)), & \aleph\in(Ƶ\cap B)\cap(Ƶ\cap Ş)=Ƶ\cap B\cap Ş \end{cases}$$

Here, if we consider Ƶ-B in the function N, since Ƶ-B=Ƶ∩B', then if an element is in the complement of B, that element is either in Ş-B or in (B∪ Ş)'. From here, if $\aleph\in$Ş-B, then $\aleph\in Ƶ\cap B'\cap Ş$ or $\aleph\in Ƶ\cap B'\cap Ş'$, hence we see that N=T with the condition $Ƶ'\cap B\cap Ş=Ƶ\cap B'\cap Ş=Ƶ\cap B\cap Ş'=\emptyset$. It is obvious that the condition $Ƶ'\cap B\cap Ş=Ƶ\cap B'\cap Ş=\emptyset$ is equivalent to the condition $(Ƶ\Delta B)\cap Ş$.

2)If $(Ƶ\Delta B)\cap Ş=Ƶ\cap B\cap Ş'=\emptyset$, then $(F,Ƶ) \overset{*}{\cup_{\varepsilon}} [(G,B) \overset{*}{\underset{\cup}{\sim}} (H,Ş)]=[(F,Ƶ) \overset{*}{\cup_{\varepsilon}} (G,B)] \overset{*}{\underset{\cup}{\sim}} [(F,Ƶ) \overset{*}{\cup_{\varepsilon}} (H,Ş)]$.

3) If $(Ƶ\Delta B)\cap Ş=Ƶ\cap B\cap Ş'=\emptyset$, then $(F,Ƶ) \overset{*}{\cup_{\varepsilon}} [(G,B) \overset{*}{\underset{*}{\sim}} (H,Ş)]=[(F,Ƶ) \overset{*}{\lambda_{\varepsilon}} (G,B)] \overset{*}{\underset{\cup}{\sim}} [(F,Ƶ) \overset{*}{\lambda_{\varepsilon}} (H,Ş)]$.

4) If $(Ƶ\Delta B)\cap Ş= Ƶ\cap B\cap Ş'=\emptyset$, then $(F,Ƶ) \overset{*}{\cup_{\varepsilon}} [(G,B) \overset{*}{\underset{\theta}{\sim}} (H,Ş)]=[(F,Ƶ) \overset{*}{\lambda_{\varepsilon}} (G,B)] \overset{*}{\underset{\cap}{\sim}} [(F,Ƶ) \overset{*}{\lambda_{\varepsilon}} (H,Ş)]$.

ii) RHS Distributions of Complementary Extended Union Operations over Complementary Soft Binary Piecewise Soft Set Operations

1)$(Ƶ\Delta B)\cap Ş=Ƶ\cap B\cap Ş'=\emptyset$ if $(F,Ƶ) \overset{*}{\underset{\cup}{\sim}} (G,B)] \overset{*}{\cup_{\varepsilon}} (H,Ş)=[(F,Ƶ) \overset{*}{\cup_{\varepsilon}} (H,Ş)] \overset{*}{\underset{\cup}{\sim}} [(G,B) \overset{*}{\cup_{\varepsilon}} (H,Ş)]$.

2) If $(Ƶ\Delta B)\cap Ş=Ƶ\cap B\cap Ş'=\emptyset$, then $(F,Ƶ) \overset{*}{\underset{\cap}{\sim}} (G,B)] \overset{*}{\cup_{\varepsilon}} (H,Ş) =[(F,Ƶ) \overset{*}{\cup_{\varepsilon}} (H,Ş)] \overset{*}{\underset{\cap}{\sim}} [(G,B) \overset{*}{\cup_{\varepsilon}} (H,Ş)]$.

3) If $(Ƶ\Delta B)\cap Ş = Ƶ\cap B\cap Ş' = \emptyset$, then $(F,Ƶ)\overset{*}{\underset{\theta}{\sim}}(G,B)]\overset{*}{\cup}_{\varepsilon}(H,Ş) = [(F,Ƶ)\overset{*}{+}_{\varepsilon}(H,Ş)]\overset{*}{\underset{\cap}{\sim}}[(G,B)\overset{*}{+}_{\varepsilon}(H,Ş)]$.

4) If $(Ƶ\Delta B)\cap Ş = Ƶ\cap B\cap Ş' = \emptyset$, then $[(F,Ƶ)\overset{*}{\underset{*}{\sim}}(G,B)]\overset{*}{\cup}_{\varepsilon}(H,Ş) = [(F,Ƶ)\overset{*}{+}_{\varepsilon}(H,Ş)]\overset{*}{\underset{\cup}{\sim}}[(G,B)\overset{*}{+}_{\varepsilon}(H,Ş)]$.

Note 4.5.1. If we consider the distributions in Theorem 4.5 and the conditions under which they are satisfied, it is obvious that the following distributions are satisfied in the set $S_{Ƶ}(U)$ without any conditions, where Ƶ is a fixed subset of E.

- $(F,Ƶ)\overset{*}{\cup}_{\varepsilon}[(G,Ƶ)\overset{*}{\underset{\cap}{\sim}}(H,Ƶ)] = [(F,Ƶ)\overset{*}{\cup}_{\varepsilon}(G,Ƶ)]\overset{*}{\underset{\cap}{\sim}}[(F,Ƶ)\overset{*}{\cup}_{\varepsilon}(H,Ƶ)]$.
- $[(F,Ƶ)\overset{*}{\underset{\cap}{\sim}}(G,Ƶ)]\overset{*}{\cup}_{\varepsilon}(H,Ƶ) = [(F,Ƶ)\overset{*}{\cup}_{\varepsilon}(H,Ƶ)]\overset{*}{\underset{\cap}{\sim}}[(G,Ƶ)\overset{*}{\cup}_{\varepsilon}(H,Ƶ)]$.
- $(F,Ƶ)\overset{*}{\cup}_{\varepsilon}[(G,Ƶ)\overset{*}{\underset{\cup}{\sim}}(H,Ƶ)] = [(F,Ƶ)\overset{*}{\cup}_{\varepsilon}(G,Ƶ)]\overset{*}{\underset{\cup}{\sim}}[(F,Ƶ)\overset{*}{\cup}_{\varepsilon}(H,Ƶ)]$.
- $[(F,Ƶ)\overset{*}{\underset{\cup}{\sim}}(G,Ƶ)]\overset{*}{\cup}_{\varepsilon}(H,Ƶ) = [(F,Ƶ)\overset{*}{\cup}_{\varepsilon}(H,Ƶ)]\overset{*}{\underset{\cup}{\sim}}[(G,Ƶ)\overset{*}{\cup}_{\varepsilon}(H,Ƶ)]$.

Theorem 4.5.2. $(S_{Ƶ}(U), \overset{*}{\underset{\cup}{\sim}}, \overset{*}{\cup}_{\varepsilon})$ is a commutative, idempotent semiring without zero but with unity.

Theorem 4.5.3 $(S_{Ƶ}(U), \overset{*}{\underset{\cap}{\sim}}, \overset{*}{\cup}_{\varepsilon})$ is a commutative, idempotent hemiring with unity.

Theorem 4.5.4. $(S_{Ƶ}(U), U_{Ƶ}, \emptyset_{Ƶ}, \overset{*}{\underset{\cap}{\sim}}, \overset{*}{\cup}_{\varepsilon})$ Bool Algebra and De Morgan Algebra.

5. Discussion

In this paper, we introduced the complementary extended union operation, and showed that the collection of all soft sets with a fixed parameter set together with the complementary extended union operation and also with other certain types of soft set operations form many important algebraic structures such as semiring, hemiring, Boolean Algebra, De Morgan Algebra. Let $S_{Ƶ}(U)$ be the collection of all soft sets over U with the fixed parameter set Ƶ, where $Ƶ \subseteq E$. Then,

- $(S_{Ƶ}(U), \overset{*}{\cup}_{\varepsilon})$ is a commutative, idempotent monoid, that is, a bounded semilattice, whose identity element is $\emptyset_{Ƶ}$.
- $(S_{E}(U), \overset{*}{\cup}_{\varepsilon})$ is a groupoid.
- $(S_{Ƶ}(U), r, \overset{*}{\cup}_{\varepsilon}, \emptyset_{Ƶ})$ is an MV-algebra.
- $(S_{Ƶ}(U), \cup_R, \overset{*}{\cup}_{\varepsilon})$, $(S_{Ƶ}(U), \cup_{\varepsilon}, \overset{*}{\cup}_{\varepsilon})$, $(S_{Ƶ}(U), \overset{*}{\cup}_{\varepsilon}, \overset{*}{\cup}_{\varepsilon})$, $(S_{Ƶ}(U), \overset{\sim}{\cup}, \overset{*}{\cup}_{\varepsilon})$, $(S_{Ƶ}(U), \overset{*}{\underset{\cup}{\sim}}, \overset{*}{\cup}_{\varepsilon})$ are commutative, idempotent semirings without zero but with unity.

- $(S_Z(U), \cap_R, \overset{*}{\cup}_\varepsilon)$, $(S_Z(U), \cap_\varepsilon, \overset{*}{\cup}_\varepsilon)$, $(S_Z(U), \overset{*}{\cap}_\varepsilon, \overset{*}{\cup}_\varepsilon)$, $(S_Z(U), \tilde{\cap}, \overset{*}{\cup}_\varepsilon)$, $(S_Z(U), \underset{\cap}{\overset{*}{\sim}}, \overset{*}{\cup}_\varepsilon)$ are commutative, idempotent hemirings with unity.
- $(S_Z(U), U_Z, \emptyset_Z, \cap_R, \overset{*}{\cup}_\varepsilon)$, $(S_Z(U), U_Z, \emptyset_Z, \cap_\varepsilon \overset{*}{\cup}_\varepsilon)$, $(S_Z(U), U_Z, \emptyset_Z, \overset{*}{\cap}_\varepsilon, \overset{*}{\cup}_\varepsilon)$, $(S_Z(U), U_Z, \emptyset_Z, \tilde{\cap}, \overset{*}{\cup}_\varepsilon)$, $(S_Z(U), U_Z, \emptyset_Z, \underset{\cap}{\overset{*}{\sim}}, \overset{*}{\cup}_\varepsilon)$ are Boolean Algebras and De Morgan Algebras.

6. Conclusion

Soft set operations are the foundational elements of soft set theory, crucial for its advancement in both theoretical and practical realms. Since its inception, numerous restricted and extended operations have been introduced for soft sets. However, this study introduces and explores the algebraic properties of a new soft set operation, which we call “the complementary extended union operation”, specifically comparing it to the union operation in classical set theory. We examine the distribution of the complementary extended union operation over various other soft set operations. By considering the distribution rules and algebraic properties of these operations, we provide an in-depth analysis of the algebraic structures formed by soft sets using this new operation. We demonstrate that the set of all soft sets with a fixed parameter set, along with the complementary extended union operation and other specific soft set operations, form many significant algebraic structures, including semirings, hemirings, Boolean algebras, and De Morgan algebras. As the concepts related to soft set operations are as vital to soft sets as basic operations are to classical set theory, examining their algebraic structures in relation to new soft set operations enhances our understanding of their applications and introduces new examples of algebraic structures. We believe this work contributes to the literature on both classical algebra and soft set theory. Future studies may explore different types of complementary extended soft set operations, along with their distributions and properties, to further identify the algebraic structures formed within the collection of soft sets with a fixed parameter set. Additionally, we think that this study will inspire researchers to propose new encryption methods based on soft sets, and thus it will serve as a foundation for various applications, particularly in decision-making and cryptography as soft sets are a powerful mathematical tool for detecting uncertain objects. Furthermore, the algebraic properties of soft algebraic structures can be re-examined and further developed in the sense of the soft set operation defined in this paper.

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Conflict of Interest

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